

Asymptotic behaviour of the fourth Painlevé transcendents in the space of initial values

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ABSTRACT

We study the asymptotic behaviour of solutions of the fourth Painlevé equation as the independent variable goes to infinity in its space of (complex) initial values, which is a generalisation of phase space described by Okamoto. We show that the limit set of each solution is compact and connected and, moreover, that any non-special solution has an infinite number of poles and infinite number of zeroes.

1. Introduction

We study the dynamics of solutions of the fourth Painlevé equation

$$P_{IV} : \frac{d^2y}{dx^2} = \frac{1}{2y} \left(\frac{dy}{dx} \right)^2 + \frac{3}{2}y^3 + 4xy^2 + 2(x^2 - \alpha)y + \frac{\beta}{y}, \quad (1.1)$$

where $y = y(x)$ is a function of $x \in \mathbb{C}$, and α, β complex constants, in the singular limit as $|x| \rightarrow \infty$ in the space of initial values, a generalisation of phase space first constructed in [15]. In this paper, we prove that each non-rational transcendental solution of P_{IV} has infinitely many zeroes and poles in \mathbb{C} (see Theorem 5.4).

We start by transforming P_{IV} to new coordinates that make the study of the limit $|x| \rightarrow \infty$ more explicit. The proof contains three ingredients: (i) the resolution of singularities of the Painlevé vector field in the space of initial values; (ii) an analytic study of the flow of the Painlevé vector field close to the exceptional lines in the resolved space; and (iii) construction of the complex limit set of each solution. Using (i) and (ii), we prove that a certain set, called the infinity set, acts as a repeller of the Painlevé flow in Okamoto's space as $|x| \rightarrow \infty$ (see Theorem 4.9). Based on the estimates in the proof of this result, we show that the limit set of solutions is non-empty, compact, connected, and invariant under the flow of the associated autonomous system (see Theorem 5.1). Then by showing that the flow intersects infinitely often with the last three exceptional lines in the space of initial values, we prove Theorem 5.4. Earlier papers by one of us provided analogous results for the first and second Painlevé equations ([5, 9]).

The fourth Painlevé equation has been studied from various perspectives: see e.g., [16, 1, 13, 14, 10, 4, 7, 8, 17, 11]. However, the study of asymptotic behaviours in the limit $|x| \rightarrow \infty$ for $x \in \mathbb{C}$ appears to be incomplete in the literature. In this paper, we provide global information

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about the solutions' limiting behaviours in the complex plane in this singular limit.

In Section 2 we describe the construction of Okamoto's space of initial values for Equation (1.1). Basic steps of the resolution procedure are given there, but details of the calculations appear in Appendix A. Section 3 is devoted to the special solutions of the fourth Painlevé equation and their relation with singular curves in the elliptic pencil underlying the autonomous system. Section 4 contains the results on asymptotic behaviour of the solutions and contains the proof of Theorem 4.9, while Section 5 provides information about limit sets and contains the proofs of Theorems 5.1 and 5.4.

2. Space of Initial Values of P_{IV}

The fourth Painlevé equation (1.1) is equivalent to the following system:

$$\begin{aligned}\frac{dy_1}{dx} &= -y_1(y_1 + 2y_2 + 2x) - 2\alpha_1, \\ \frac{dy_2}{dx} &= y_2(2y_1 + y_2 + 2x) - 2\alpha_2,\end{aligned}\tag{2.1}$$

with $y = y_1$, $\alpha = 1 - \alpha_1 - 2\alpha_2$, $\beta = -2\alpha_1^2$. System (2.1) is Hamiltonian with the following Hamiltonian function:

$$H(x, y_1, y_2) = -y_1 y_2 (y_1 + y_2 + 2x) + 2\alpha_2 y_1 - 2\alpha_1 y_2,\tag{2.2}$$

that is, (2.1) is equivalent to Hamilton's equations of motion

$$\frac{dy_1}{dx} = \frac{\partial H}{\partial y_2}, \quad \frac{dy_2}{dx} = -\frac{\partial H}{\partial y_1}.$$

The asymptotic behaviour of the Painlevé transcendents was first studied by Boutroux [2, 3]. There, for the first Painlevé equation, he made certain change of variables in order to make the asymptotic behaviours more explicit. In the same spirit, we make the following change of variables for (2.1):

$$y_1 = xu, \quad y_2 = xv, \quad z = \frac{x^2}{2}$$

which transforms the system (2.1) to

$$\begin{aligned}u' &= -u(u + 2v + 2) - \frac{\alpha_1}{z} - \frac{u}{2z}, \\ v' &= v(2u + v + 2) - \frac{\alpha_2}{z} - \frac{v}{2z}.\end{aligned}\tag{2.3}$$

Here and later in this paper, primes denote differentiation with respect to z .

For each $z \neq 0$, and each $(u_0, v_0) \in \mathbb{C}^2$, there is a unique solution of (2.3) satisfying the initial conditions $u(z_0) = u_0$, $v(z_0) = v_0$. Since the solutions are meromorphic and therefore will become unbounded in neighbourhoods of movable poles, it is natural to consider the solutions as maps from \mathbb{C} to \mathbb{CP}^2 . However, for any given $z_0 \neq 0$, infinitely many solutions may pass through certain points in \mathbb{CP}^2 . Such points will be called *base points* in this paper.

To resolve the flow through such points, we need to construct the *space of initial conditions* (see [6]), where the graph of each solution will represent a separate leaf of the foliation. The spaces of initial conditions for all six Painlevé equations were constructed in [15]. The solutions are separated by resolving (i.e., blowing up) the base points.

In this paper, we explicitly construct such a resolution of the system (2.3). The details of the

calculation can be found in Appendix A, and now we describe the main steps in that resolution process.

2.1 Resolution of singularities

System (2.3) has no singularities in the affine part of \mathbb{CP}^2 . However, at the line \mathcal{L}_0 at the infinity, as calculated in Appendix A.1, the system has three base points: b_0, b_1, b_2 , whose coordinates do not depend on z .

In the next step, we construct blow ups at points b_0, b_1, b_2 . In the resulting space, we obtain three exceptional lines which we denote by $\mathcal{L}_1, \mathcal{L}_2, \mathcal{L}_3$ respectively. The induced flow will have one base point on each of these lines, denote them by b_3, b_4, b_5 respectively. These points are also base points for the autonomous system, as their coordinates do not depend on z . See Appendix A.2 for details.

Next, blow ups at points b_3, b_4, b_5 are constructed. The corresponding exceptional lines are $\mathcal{L}_4, \mathcal{L}_5, \mathcal{L}_6$. On each of these three lines, there is a base point of the flow. We denote them by b_6, b_7, b_8 . The coordinates of these points depend on z and they approach the base points of the autonomous flow as $z \rightarrow \infty$. See Appendix A.3 for details.

Finally, blow ups at b_6, b_7, b_8 show that there are no new base points. The exceptional lines are denoted by $\mathcal{L}_7(z), \mathcal{L}_8(z), \mathcal{L}_9(z)$.

By this procedure, we constructed the fibers $\mathcal{F}(z)$, $z \in \mathbb{C} \cup \{\infty\} \setminus \{0\}$ of the Okamoto space \mathcal{O} for the system (2.3), see Figure 1. We denote by \mathcal{L}_i^* the proper preimages of the lines \mathcal{L}_i , $0 \leq i \leq 6$.

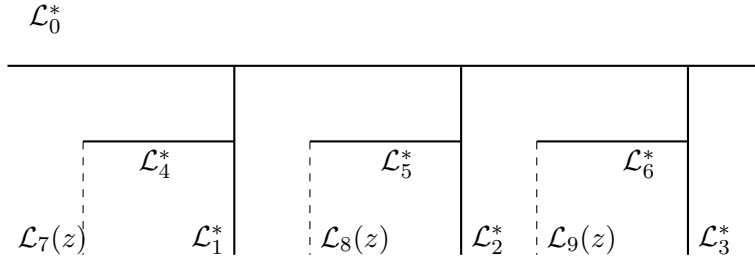


FIGURE 1. Fiber $\mathcal{F}(z)$ of the Okamoto space. At the points of $\mathcal{L}_7(z)$, u has a pole and v a zero; on $\mathcal{L}_8(z)$ both have poles; and on $\mathcal{L}_9(z)$, u has a zero and v a pole.

The set where the vector field associated to (2.1) becomes infinite will be denoted $\mathcal{I} = \bigcup_{j=0}^6 \mathcal{L}_j^*$.

2.2 The autonomous system

The fiber $\mathcal{F}(\infty)$ of the Okamoto space corresponds to the autonomous system obtained by omitting the z -dependent terms in (2.3):

$$\begin{aligned} u' &= -u(u + 2v + 2), \\ v' &= v(2u + v + 2), \end{aligned} \tag{2.4}$$

which is equivalent to

$$u'' = \frac{u'^2}{2u} + \frac{3}{2}u^3 + 4u^2 + 2u,$$

and further to the following family:

$$(u')^2 = \frac{1}{2}u^4 + 2u^3 + 2u^2 + cu, \quad c \in \mathbb{C}.$$

The solutions of (2.4) are thus elliptic functions.

System (2.4) is Hamiltonian, i.e.,

$$u' = \frac{\partial E}{\partial v}, \quad v' = -\frac{\partial E}{\partial u}.$$

where $E = -uv(u + v + 2)$.

Note that b_0, \dots, b_5 are base points of (2.4) as well, while b_6, b_7, b_8 will tend to the base points of the autonomous system as $z \rightarrow \infty$.

3. The special solutions

In this section, we analyse singular cubic curves in the pencil parametrised by solutions of the autonomous system (2.4) and show that the corresponding solutions of P_{IV} are either rational or given by parabolic cylinder and exponential functions.

3.1 Special solutions and singular cubic curves

The pencil of elliptic curves arising from the Hamiltonian of the autonomous system (2.4) is given by the zero set of $h(u, v) = c + uv(u + v + 2)$. For general values of constant c , the corresponding curves will be smooth. To investigate singularities, consider the conditions

$$\frac{\partial h}{\partial u} = 0, \quad \frac{\partial h}{\partial v} = 0,$$

which give

$$v(2u + v + 2) = 0, \quad u(u + 2v + 2) = 0.$$

The solutions are $(0, 0)$, $(0, -2)$, $(-2, 0)$, which lie on the curve corresponding to $c = 0$, and $(-\frac{2}{3}, -\frac{2}{3})$ on the curve corresponding to $c = -\frac{8}{27}$. In other words, there are two singular curves in the pencil and the first (given by $c = 0$) contains three singular points, while the second (given by $c = -\frac{8}{27}$) contains one singularity.

Consider first the case $c = 0$. The corresponding curve is $uv(u + v + 2) = 0$, which is a singular cubic consisting of three lines: $u = 0$, $v = 0$ and $u + v + 2 = 0$.

PROPOSITION 3.1. *For (u, v) being a solution of the non-autonomous system (2.3), each derivative of $E = -uv(u + v + 2)$ with respect to z vanishes if any of the following three sets of conditions is satisfied:*

- (i) $u = 0$ and $\alpha_1 = 0$;
- (ii) $v = 0$ and $\alpha_2 = 0$;
- (iii) $u + v + 2 = 0$ and $\alpha_1 + \alpha_2 = 1$.

Proof. We have:

$$\begin{aligned} E' &= \frac{dE}{dz} = \frac{\partial E}{\partial u}u' + \frac{\partial E}{\partial v}v' \\ &= \frac{1}{z} \left(\alpha_2 u(u + 2v + 2) + \alpha_1 v(2u + v + 2) - uv - \frac{3E}{2} \right). \end{aligned}$$

Case 1. Note that u is a divisor of E , which is polynomial in u and v , and that, when $\alpha_1 = 0$, u is also a divisor of u' . By induction, it follows that all derivatives of E will be multiples of u and polynomials of u, v and thus equal to zero for $u = 0$.

Case 2. The proof is analogous to that of Case 1.

Case 3. Note that that E is a product of $u + v + 2$ and a polynomial of u, v . For $\alpha_1 + \alpha_2 = 1$, the derivative of $u + v + 2$ is of the same form:

$$(u + v + 2)' = -\frac{(u + v + 2)(1 + 2uz - 2vz)}{2z}.$$

By induction, the same result as in Cases 1 and 2 will hold for all derivatives of E . \square

REMARK 3.2. $\alpha_1 = 0$ is equivalent to $\beta = 0$; $\alpha_2 = 0$ to $\beta = -2(1 - \alpha)^2$; and $\alpha_1 + \alpha_2 = 1$ to $\beta = -2(1 + \alpha)^2$.

For $\beta = -2(1 + \epsilon\alpha)^2$, $\epsilon \in \{-1, 1\}$, the Painlevé equation (1.1) is equivalent to the following Riccati equation:

$$\frac{dy}{dx} = \epsilon(y^2 + 2xy) - 2(1 + \epsilon\alpha),$$

which can be solved in terms of parabolic cylinder and exponential functions:

$$y = -\epsilon \frac{d\phi/dx}{\phi},$$

$$\phi(x) = \left(C_1 U\left(\alpha + \frac{\epsilon}{2}, \sqrt{2}x\right) + C_2 V\left(\alpha + \frac{\epsilon}{2}, \sqrt{2}x\right) \right) e^{\epsilon x^2/2}.$$

Note that a zero of $\phi(x)$ corresponds to a pole of $u(z)$.

For $\epsilon = 1$, which is Case 3 of Proposition 3.1, v also has a pole, thus each point of \mathcal{L}_8^* corresponds to a special solution. For $\epsilon = -1$, which is Case 2 of Proposition 3.1, v has a zero, thus each point of \mathcal{L}_7^* corresponds to a special solution. For $\beta = 0$, which is Case 1 of Proposition 3.1, the solution can be expressed in terms of Hermite polynomials. Each point of \mathcal{L}_9^* corresponds to such a solution.

Now, consider the case $c = -\frac{8}{27}$. The corresponding curve is $uv(u + v + 2) = \frac{8}{27}$ and it has a unique singular point $(-\frac{2}{3}, -\frac{2}{3})$. For $\tilde{u} = u + \frac{2}{3}$ and $\tilde{v} = v + \frac{2}{3}$, the equation of the curve becomes:

$$-\frac{2}{3}(\tilde{u}^2 + \tilde{u}\tilde{v} + \tilde{v}^2) + \tilde{u}\tilde{v}(\tilde{u} + \tilde{v}) = 0,$$

thus the curve has an ordinary self-intersection at the singular point. The corresponding solutions are rational.

3.2 Special rational solutions of P_{IV}

Consider the following rational solutions of P_{IV} :

$$\begin{aligned} y &= \pm \frac{1}{x}, & \text{for } \alpha = \pm 2, \beta = -2; \\ y &= -2x, & \text{for } \alpha = 0, \beta = -2; \\ y &= -\frac{2}{3}x, & \text{for } \alpha = 0, \beta = -\frac{2}{9}. \end{aligned}$$

The corresponding solutions of the system (2.3) are:

$$\begin{aligned}
 u &= \frac{1}{2z}, & v &= 0, & (\alpha_1, \alpha_2) &= (-1, 0); \\
 u &= -\frac{1}{2z}, & v &= \frac{1}{2z}, & (\alpha_1, \alpha_2) &= (1, 1); \\
 u &= \frac{1}{2z}, & v &= -2, & (\alpha_1, \alpha_2) &= (1, -1); \\
 u &= -\frac{1}{2z}, & v &= \frac{1}{2z} - 2, & (\alpha_1, \alpha_2) &= (-1, 2); \\
 u &= -2, & v &= 0, & (\alpha_1, \alpha_2) &= (1, 0); \\
 u &= -2, & v &= -\frac{1}{2z}, & (\alpha_1, \alpha_2) &= (-1, 1); \\
 u &= -\frac{2}{3}, & v &= -\frac{2}{3}, & (\alpha_1, \alpha_2) &= \left(\frac{1}{3}, \frac{1}{3}\right); \\
 u &= -\frac{2}{3}, & v &= -\frac{1}{2z} - \frac{2}{3}, & (\alpha_1, \alpha_2) &= \left(-\frac{1}{3}, \frac{2}{3}\right).
 \end{aligned}$$

All other rational solutions can be obtained from these solutions by Bäcklund transformations [12]:

$$\begin{aligned}
 s_1 &: (u, v; \alpha_1, \alpha_2) \rightarrow (u, v + \frac{\alpha_1}{zu}; -\alpha_1, \alpha_1 + \alpha_2), \\
 s_2 &: (u, v; \alpha_1, \alpha_2) \rightarrow (u - \frac{\alpha_2}{zv}, v; \alpha_1 + \alpha_2, -\alpha_2), \\
 s_3 &: (u, v; \alpha_1, \alpha_2) \rightarrow (u - \frac{\alpha_3}{z(u+v+2)}, v + \frac{\alpha_3}{z(u+v+2)}; 1 - \alpha_2, 1 - \alpha_1), \\
 \pi &: (u, v; \alpha_1, \alpha_2) \rightarrow (v, -2 - u - v; \alpha_2, 1 - \alpha_1 - \alpha_2),
 \end{aligned}$$

which have the following properties:

$$\begin{aligned}
 s_1^2 &= s_2^2 = s_3^2 = 1, & (s_1 s_2)^3 &= (s_2 s_3)^3 = (s_3 s_1)^3 = 1, & \pi^3 &= 1, \\
 s_2 &= \pi s_1 \pi^2, & s_3 &= \pi s_0 \pi^2, & s_0 &= \pi s_3 \pi^2.
 \end{aligned}$$

Also, all special solutions of the fourth Painlevé equation can be obtained by the Bäcklund transformations from the solutions mentioned in Section 3.1.

4. The solutions near the infinity set

In this section, we will study the behaviour of the solutions of the system (2.1) near the set \mathcal{I} , where the vector field associated to the system is infinite.

In Lemmas 4.1-4.8 and Theorem 4.9, we prove that \mathcal{I} is repelling, i.e. the solutions do not intersect it; and, moreover, each solution approaching sufficiently close to \mathcal{I} at point z will have a pole in a neighbourhood of z .

LEMMA 4.1. *For every $\varepsilon > 0$ there exists a neighbourhood U of \mathcal{L}_0^* such that*

$$\left| \frac{E'}{E} + \frac{3}{2z} \right| < \varepsilon \quad \text{in } U.$$

For each compact subset K of $(\mathcal{L}_1^ \setminus \mathcal{L}_4^*) \cup (\mathcal{L}_2^* \setminus \mathcal{L}_5^*) \cup (\mathcal{L}_3^* \setminus \mathcal{L}_6^*)$, there exists a neighbourhood V*

of K and a constant $C > 0$ such that:

$$\left| z \frac{E'}{E} \right| < C \quad \text{in } V \text{ for all } z \neq 0.$$

Proof. In the charts (u_{02}, v_{02}) and (u_{03}, v_{03}) (see Appendix A.1), the function

$$r = \frac{E'}{E} + \frac{3}{2z}$$

is equal to:

$$\begin{aligned} r_{02} &= -\frac{u_{02}(\alpha_2 + 2\alpha_2 u_{02} + (2\alpha_1 + 2\alpha_2 - 1)v_{02} + 2\alpha_1 u_{02} v_{02} + \alpha_1 v_{02}^2)}{v_{02}(1 + 2u_{02} + v_{02})z}, \\ r_{03} &= -\frac{u_{03}(\alpha_1 + 2\alpha_1 u_{03} + (2\alpha_1 + 2\alpha_2 - 1)v_{03} + 2\alpha_2 u_{03} v_{03} + \alpha_2 v_{03}^2)}{v_{03}(1 + 2u_{03} + v_{03})z}. \end{aligned}$$

The first statement of the lemma follows immediately from these expressions, since \mathcal{L}_0^* is given by $u_{02} = 0$ and $u_{03} = 0$ in those charts, see Section A.1.

Near \mathcal{L}_1^* , in the respective coordinate charts (see Section A.2), we have

$$z \frac{E'}{E} + \frac{3}{2} \sim \begin{cases} -\alpha_2 u_{11} \\ -\frac{\alpha_2}{v_{12}} \end{cases}$$

Since \mathcal{L}_4^* is given by $v_{12} = 0$, see Section A.3, the statement of the lemma is true for the compact sets K contained in a neighbourhood of $\mathcal{L}_1^* \setminus \mathcal{L}_4^*$.

On \mathcal{L}_2^* , (see Section A.2), we have

$$z \frac{E'}{E} \sim \begin{cases} -\frac{3 + 2(2 + \alpha_1 + \alpha_2)u_{21}}{2(1 + 2u_{21})} \\ -\frac{4 + 2\alpha_1 + 2\alpha_2 + 3v_{22}}{2(2 + v_{22})} \end{cases}$$

Therefore, since \mathcal{L}_5^* is given by the equations $u_{21} = -\frac{1}{2}$ and $v_{22} = -2$, the statement is true for the compacts contained in a neighbourhood of $\mathcal{L}_2^* \setminus \mathcal{L}_5^*$.

On \mathcal{L}_3^* , (see Section A.2), we have

$$z \frac{E'}{E} + \frac{3}{2} \sim \begin{cases} -\alpha_1 u_{31} \\ -\frac{\alpha_1}{v_{32}} \end{cases}$$

Since \mathcal{L}_6 is given by $v_{32} = 0$, the statement of the lemma is true for the compact sets K contained in a neighbourhood of $\mathcal{L}_3^* \setminus \mathcal{L}_6^*$. \square

LEMMA 4.2. *There exists a continuous complex valued function d on a neighbourhood of the infinity set \mathcal{I} in the Okamoto space, such that:*

$$d = \begin{cases} \frac{1}{E}, & \text{in a neighbourhood of } \mathcal{I} \setminus (\mathcal{L}_4^* \cup \mathcal{L}_5^* \cup \mathcal{L}_6^*), \\ J_{71}, & \text{in a neighbourhood of } \mathcal{L}_4^* \setminus \mathcal{L}_1^*, \\ \frac{J_{82}}{2}, & \text{in a neighbourhood of } \mathcal{L}_5^* \setminus \mathcal{L}_2^*, \\ -J_{91}, & \text{in a neighbourhood of } \mathcal{L}_6^* \setminus \mathcal{L}_3^*. \end{cases}$$

Proof. Assume d is defined by $\frac{1}{E}$, in a neighbourhood of $\mathcal{I} \setminus (\mathcal{L}_4^* \cup \mathcal{L}_5^* \cup \mathcal{L}_6^*)$. From Section A.4, we have that the line \mathcal{L}_4^* is determined by $u_{71} = 0$ in the (u_{71}, v_{71}) chart. Thus as we approach

\mathcal{L}_4^* , i.e., as $u_{71} \rightarrow 0$, we have

$$EJ_{71} \sim 1 + \frac{\alpha_2}{v_{71}z}$$

which provides the second result.

From Section A.4, we have that the line \mathcal{L}_5^* is given by $v_{82} = 0$ in the (u_{82}, v_{82}) chart. Thus as we approach \mathcal{L}_5^* :

$$EJ_{82} \sim 2 - \frac{1 - \alpha_1 - \alpha_2}{4u_{82}z}$$

which gives the third result.

From Section A.4, we have that the line \mathcal{L}_6^* is given by $u_{91} = 0$ in the (u_{91}, v_{91}) chart. Then as we approach \mathcal{L}_6^* :

$$EJ_{91} \sim -1 + \frac{\alpha_1}{v_{91}z}$$

which provides the fourth result. \square

LEMMA 4.3 Behaviour near $\mathcal{L}_4^* \setminus \mathcal{L}_1^*$. *If a solution at the complex time z is sufficiently close to $\mathcal{L}_4^* \setminus \mathcal{L}_1^*$, then there exists a unique $\zeta \in \mathbb{C}$ such that:*

- (i) $v_{71}(\zeta) = 0$, i.e. $\zeta \in \mathcal{L}_7$;
- (ii) $|z - \zeta| = O(|d(z)||v_{71}(z)|)$ for small $d(z)$ and bounded $|v_{71}(z)|$.

In other words, the solution has a pole at $z = \zeta$.

For large $R_4 > 0$, consider the set $\{z \in \mathbb{C} \mid |v_{71}| \leq R_4\}$. Then, its connected component containing ζ is an approximate disk D_4 with centre ζ and radius $|d(\zeta)|R_4$, and $z \mapsto v_{71}(z)$ is a complex analytic diffeomorphism from that approximate disk onto $\{v \in \mathbb{C} \mid |v| \leq R_4\}$.

Proof. For the study of the solutions near $\mathcal{L}_4^* \setminus \mathcal{L}_1^*$, we use the coordinates (u_{71}, v_{71}) . In this chart, the line $\mathcal{L}_4^* \setminus \mathcal{L}_1^*$ is given by the equation $u_{71} = 0$ and parametrized by $v_{71} \in \mathbb{C}$ (see Section A.4). Moreover, \mathcal{L}_7^* (without one point) is given by $v_{71} = 0$ and parametrized by $u_{71} \in \mathbb{C}$. (Equivalent arguments in the alternative chart (u_{72}, v_{72}) cover the missing point of \mathcal{L}_7^* .)

Asymptotically, for $u_{71} \rightarrow 0$, and bounded v_{71} , $1/z$, we have

$$v'_{71} \sim \frac{1}{u_{71}}, \tag{4.1a}$$

$$J_{71} = -u_{71}, \tag{4.1b}$$

$$\frac{J'_{71}}{J_{71}} = 2 + \frac{3}{2z} + O(u_{71}) = 2 + \frac{3}{2z} + O(J_{71}), \tag{4.1c}$$

$$EJ_{71} \sim 1 + \frac{\alpha_2}{z v_{71}}. \tag{4.1d}$$

Integrating (4.1c) from ζ to z , we get

$$J_{71}(z) = J_{71}(\zeta) e^{2(z-\zeta)} \left(\frac{z}{\zeta}\right)^{3/2} (1 + o(1)), \quad \text{for } \frac{z}{\zeta} \sim 1.$$

Hence, using Equation (4.1b), u_{71} is approximately equal to a small constant, and from (4.1a) it follows that:

$$v_{71}(z) \sim v_{71}(\zeta) + \frac{z - \zeta}{u_{71}}.$$

Thus, if z runs over an approximate disk D centred at ζ with radius $|u_{71}|R$, then v_{71} fills an approximate disk centred at $v_{71}(\zeta)$ with radius R . Therefore, if $u_{71}(\zeta) \ll 1/\zeta$, for $z \in D$, the

solution satisfies

$$\frac{u_{71}(z)}{u_{71}(\zeta)} \sim 1,$$

and $v_{71}(z)$ is a complex analytic diffeomorphism from D onto an approximate disk with centre $v_{71}(\zeta)$ and radius R . If R is sufficiently large, we will have $0 \in v_{71}(D)$, i.e. the solution of the Painlevé equation will have a pole at a unique point in D .

Now, it is possible to take ζ to be the pole point. For $|z - \zeta| \ll |\zeta|$, we have:

$$\frac{d(z)}{d(\zeta)} \sim 1, \quad \text{i.e.} \quad -\frac{u_{71}(z)}{d(\zeta)} \sim \frac{J_{71}(z)}{d(\zeta)} \sim 1,$$

and

$$v_{71}(z) \sim \frac{z - \zeta}{u_{71}} \sim -\frac{z - \zeta}{d(\zeta)}.$$

Let R_4 be a large positive real number. Then the equation $|v_{71}(z)| = R_4$ corresponds to $|z - \zeta| \sim |d(\zeta)|R_4$, which is still small compared to $|\zeta|$ if $|d(\zeta)|$ is sufficiently small. Denote by D_4 the connected component of the set of all $z \in \mathbb{C}$ such that $\{z \mid |v_{71}(z)| \leq R_4\}$ is an approximate disk with centre ζ and radius $|d(\zeta)|R_4$.

More precisely, v_{71} is a complex analytic diffeomorphism from D_4 onto $\{v \in \mathbb{C} \mid |v| \leq R_4\}$, and

$$\frac{d(z)}{d(\zeta)} \sim 1 \quad \text{for all } z \in D_4.$$

The function $E(z)$ has a simple pole at $z = \zeta$. From (4.1d), we have:

$$E(z)J_{71}(z) \sim 1 \quad \text{when} \quad 1 \gg \frac{1}{|zv_{71}(z)|} \sim \left| \frac{u_{71}(\zeta)}{\zeta(z - \zeta)} \right| = \frac{|d(\zeta)|}{|\zeta(z - \zeta)|},$$

that is, when

$$|z - \zeta| \gg \frac{|d(\zeta)|}{|\zeta|}.$$

We assume $R_4 \gg \frac{1}{|\zeta|}$ and, therefore, we have

$$|z - \zeta| \sim |d(\zeta)|R_4 \gg \frac{|d(\zeta)|}{|\zeta|}.$$

Thus $E(z)J_{71}(z) \sim 1$ for the annular disk $z \in D_4 \setminus D'_4$, where D'_4 is a disk centred at ζ with small radius compared to the radius of D_4 . \square

LEMMA 4.4 Behaviour near $\mathcal{L}_1^* \setminus \mathcal{L}_0^*$. For large finite $R_1 > 0$, consider the set of all $z \in \mathbb{C}$, such that the solution at complex time z is close to $\mathcal{L}_1^* \setminus \mathcal{L}_0^*$, with $|v_{41}(z)| \leq R_1$, but not close to \mathcal{L}_4^* . Then this set is the complement of D_4 in an approximate disk D_1 with centre at ζ and radius $\sim \sqrt{|d(\zeta)|R_1}$. More precisely, $z \mapsto v_{41}$ defines a 2-fold covering from the annular domain $D_1 \setminus D_4$ onto the complement in $\{u \in \mathbb{C} \mid |u| \leq R_1\}$ of an approximate disk with centre at the origin and small radius $\sim |d(\zeta)|R_4^2$, where $v_{41}(z) \sim -d(\zeta)(z - \zeta)^2$.

Proof. Set $\mathcal{L}_1^* \setminus \mathcal{L}_0^*$ is visible in the chart (u_{41}, v_{41}) , where it is given by the equation $u_{41} = 0$ and parametrized by $v_{41} \in \mathbb{C}$, see Section A.3. In that chart, the line \mathcal{L}_4^* (without one point) is given by the equation $v_{41} = 0$ and parametrized by $u_{41} \in \mathbb{C}$.

For $u_{41} \rightarrow 0$ and bounded v_{41} and $1/z$, we have:

$$u'_{41} \sim -\frac{1}{v_{41}}, \quad (4.2a)$$

$$v'_{41} \sim \frac{2}{u_{41}}, \quad (4.2b)$$

$$J_{41} = -u_{41}^2 v_{41}, \quad (4.2c)$$

$$EJ_{41} \sim 1, \quad (4.2d)$$

$$\frac{E'}{E} \sim -\frac{3}{2z} - \frac{\alpha_2}{v_{41}z}. \quad (4.2e)$$

From (4.2e) and (4.2a), we get:

$$\frac{E'}{E} \sim -\frac{3}{2z} + \frac{\alpha_2}{z} u'_{41}.$$

Integrating from z_0 to z_1 , we obtain:

$$\log \frac{E(z_1)}{E(z_0)} \sim \log \left(\frac{z_1}{z_0} \right)^{-3/2} + \alpha_2 \left(\frac{u_{41}(z_1)}{z_1} - \frac{u_{41}(z_0)}{z_0} + \int_{z_0}^{z_1} \frac{u_{41}(z)}{z^2} dz \right).$$

Therefore $E(z_1)/E(z_0) \sim 1$, if for all z on the segment from z_0 to z_1 we have $|z - z_0| \ll |z_0|$ and $|u_{41}(z)| \ll |z_0|$. We choose z_0 on the boundary of D_4 from the proof of Lemma 4.3. Then we have

$$\frac{d(\zeta)}{d(z_0)} \sim E(z_0)d(\zeta) \sim E(z_0)J_{71}(z_0) \sim 1 \quad \text{and} \quad |v_{71}(z_0)| = R_4,$$

which implies that

$$|u_{41}| = \left| \frac{1}{v_{71} + \frac{\alpha_2}{z}} \right| \sim \frac{1}{R_4} \ll 1.$$

Furthermore, equations (4.2c) and (4.2d) imply that:

$$|v_{41}(z_0)| = \frac{|J_{41}(z_0)|}{|u_{41}(z_0)|^2} \sim |d(\zeta)|R_4^2,$$

which is small when $|d(\zeta)|$ is sufficiently small.

Since D_4 is an approximate disk with centre ζ and small radius approximately equal to $|d(\zeta)|R_4$, and $R_4 \gg |\zeta|^{-1}$, we have that $|v_{71}(z)| \geq R_4 \gg 1$. Writing $z = \zeta + r(z_0 - \zeta)$, $r \geq 1$, where $r \geq 1$, we have $u_{41}(z) \ll 1$ and

$$\frac{|z - z_0|}{|z_0|} = (r - 1) \left| 1 - \frac{\zeta}{z_0} \right| \ll 1 \quad \text{if} \quad r - 1 \ll \frac{1}{|1 - \frac{\zeta}{z_0}|}.$$

Then equations (4.2c), (4.2d) and $E \sim d(\zeta)^{-1}$ yield

$$u_{41}^{-1} \sim \left(-\frac{v_{41}}{d(\zeta)} \right)^{1/2},$$

which in combination with (4.2b) leads to

$$\frac{d}{dz} (v_{41}^{1/2}) \sim (-d(\zeta))^{-1/2}.$$

Hence, we get

$$v_{41}^{1/2} \sim v_{41}(z_0)^{1/2} + (-d(\zeta))^{-1/2}(z - z_0),$$

and therefore

$$v_{41}(z) \sim -\frac{(z-z_0)^2}{d(\zeta)} \quad \text{if } |z-z_0| \gg |d(\zeta)v_{41}(z_0)|^{1/2}.$$

For large finite $R_1 > 0$, the equation $|v_{41}| = R_1$ corresponds to $|z-z_0| \sim \sqrt{|d(\zeta)|R_1}$, which is still small compared to $|z_0| \sim |\zeta|$, and therefore $|z-\zeta| \leq |z-z_0| + |z_0-\zeta| \ll |\zeta|$. This proves the statement of the lemma. \square

LEMMA 4.5 Behaviour near $\mathcal{L}_5^* \setminus \mathcal{L}_2^*$. *If a solution at the complex time z is sufficiently close to $\mathcal{L}_5^* \setminus \mathcal{L}_2^*$, then there exists a unique $\zeta \in \mathbb{C}$ such that:*

- (i) $u_{82}(\zeta) = 0$, i.e. $\zeta \in \mathcal{L}_8$;
- (ii) $|z-\zeta| = O(|d(z)||u_{82}(z)|)$ for small $d(z)$ and limited $|u_{82}(z)|$.

In other words, the solution has a pole at $z = \zeta$.

For large $R_5 > 0$, consider the set $\{z \in \mathbb{C} \mid |u_{82}| \leq R_5\}$. Then, its connected component containing ζ is an approximate disk D_5 with centre ζ and radius $|d(\zeta)|R_5$, and $z \mapsto u_{82}(z)$ is a complex analytic diffeomorphism from that approximate disk onto $\{u \in \mathbb{C} \mid |u| \leq R_5\}$.

Proof. For the study of solutions near $\mathcal{L}_5^* \setminus \mathcal{L}_2^*$, we use the coordinates (u_{82}, v_{82}) . The line $\mathcal{L}_5^* \setminus \mathcal{L}_2^*$ is given by the equation $v_{82} = 0$ and parametrized by $u_{82} \in \mathbb{C}$; see Section A.4. Moreover, $\mathcal{L}_8(z)$ (without one point), is given by $u_{82} = 0$ and parametrized by $v_{82} \in \mathbb{C}$. Asymptotically, for $v_{82} \rightarrow 0$, and bounded u_{82} , $1/z$, we have:

$$u'_{82} \sim \frac{8}{v_{82}}, \tag{4.3a}$$

$$J_{82} \sim \frac{v_{82}}{8}, \tag{4.3b}$$

$$\frac{J'_{82}}{J_{82}} = -2 + \frac{8-5(\alpha_1+\alpha_2)}{2z} + 24u_{82} + \frac{3(1-\alpha_1-\alpha_2)}{2zu_{82}^2}, \tag{4.3c}$$

$$EJ_{82} \sim 2 - \frac{1-\alpha_1-\alpha_2}{4u_{82}z}. \tag{4.3d}$$

Integrating (4.3c) from ζ to z , we get:

$$J_{82}(z) = J_{82}(\zeta)e^{K(z-\zeta)} \left(\frac{z}{\zeta}\right)^{(8-5(\alpha_1+\alpha_2))/2} (1+o(1)),$$

$$K = -2 + 24u_{82}(\tilde{\zeta}) + \frac{3(1-\alpha_1-\alpha_2)}{2\tilde{\zeta}u_{82}^2(\tilde{\zeta})},$$

where $\tilde{\zeta}$ is on the integration path.

Because of (4.3b), v_{82} is approximately equal to a small constant, and from (4.3a) follows that:

$$u_{82} \sim u_{82}(\zeta) + 8\frac{(z-\zeta)}{v_{82}(\zeta)}.$$

Thus, if z runs over an approximate disk D centred at ζ with radius $\frac{1}{8}|v_{82}|R$, then u_{82} fills an approximate disk centred at $u_{82}(\zeta)$ with radius R . Therefore, if $v_{82}(\zeta) \ll \zeta$, the solution has the following properties for $z \in D$:

$$\frac{v_{82}(z)}{v_{82}(\zeta)} \sim 1,$$

and u_{82} is a complex analytic diffeomorphism from D onto an approximate disk with centre $u_{82}(\zeta)$ and radius R . If R is sufficiently large, we will have $0 \in u_{82}(D)$, i.e. the solution of the Painlevé equation will have a pole at a unique point in D .

Not, it is possible to take ζ to be the pole point. For $|z - \zeta| \ll |\zeta|$, we have:

$$\begin{aligned} \frac{d(z)}{d(\zeta)} &\sim 1, \quad \text{i.e.} \quad \frac{v_{82}(z)}{16d(\zeta)} \sim \frac{J_{82}(z)}{2d(\zeta)} \sim 1, \\ u_{82}(z) &\sim \frac{8(z - \zeta)}{v_{82}} \sim \frac{z - \zeta}{2d(\zeta)}. \end{aligned}$$

Let R_5 be a large positive real number. Then the equation $|u_{82}(z)| = R_5$ corresponds to $|z - \zeta| \sim 2|d(\zeta)|R_5$, which is still small compared to $|\zeta|$ if $|d(\zeta)|$ is sufficiently small. Denote by D_5 the connected component of the set of all $z \in \mathbb{C}$ such that $\{z \mid |u_{82}(z)| \leq R_5\}$ is an approximate disk with centre ζ and radius $2|d(\zeta)|R_5$. More precisely, u_{82} is a complex analytic diffeomorphism from D_5 onto $\{u \in \mathbb{C} \mid |u| \leq R_5\}$, and

$$\frac{d(z)}{d(\zeta)} \sim 1 \quad \text{for all } z \in D_5.$$

The function $E(z)$ has a simple pole at $z = \zeta$. From (4.3d), we have:

$$E(z)J_{82}(z) \sim 2 \quad \text{when} \quad 1 \gg \frac{1}{|zu_{82}(z)|} \sim \left| \frac{v_{82}(\zeta)}{8\zeta(z - \zeta)} \right| \sim \frac{|d(\zeta)|}{|z - \zeta|},$$

that is when $|z - \zeta| \gg \frac{|d(\zeta)|}{|\zeta|}$.

Since $R_5 \ll 1/|\zeta|$, the approximate radius of D_5 is given by

$$|d(\zeta)|R_5 \gg \frac{|d(\zeta)|}{|\zeta|}$$

Thus $E(z)J_{82}(z) \sim 2$ for $z \in D_5 \setminus D'_5$, where D'_5 is a disk centred at ζ with small radius compared to the radius of D_5 . \square

LEMMA 4.6 Behaviour near $\mathcal{L}_2^* \setminus \mathcal{L}_0^*$. For large finite $R_2 > 0$, consider the set of all $z \in \mathbb{C}$, such that the solution at complex time z is close to $\mathcal{L}_2^* \setminus \mathcal{L}_0^*$, with $|u_{52}(z)| \leq R_2$, but not close to \mathcal{L}_5^* . Then that set is the complement of D_5 in an approximate disk D_2 with centre at ζ and radius $\sim \sqrt{|d(\zeta)|R_2}$. More precisely, $z \mapsto u_{52}$ defines a 2-fold covering from the annular domain $D_2 \setminus D_5$ onto the complement in $\{u \in \mathbb{C} \mid |u| \leq R_2\}$ of an approximate disk with centre at the origin and small radius $\sim |d(\zeta)|R_5^2$, where $u_{52}(z) \sim -d(\zeta)(z - \zeta)^2$.

Proof. The line \mathcal{L}_2^* is visible in the coordinate system (u_{52}, v_{52}) , where it is given by the equation $v_{52} = 0$ and parametrized by $u_{52} \in \mathbb{C}$; see Section A.3. In that chart, line \mathcal{L}_5^* without one point is given by the equation $u_{52} = 0$ and parametrized by $v_{52} \in \mathbb{C}$, while the line \mathcal{L}_0^* without one point is given by the equation $u_{52} = \frac{1}{2}$ and also parametrized by $v_{52} \in \mathbb{C}$. For $v_{52} \rightarrow 0$ and bounded

u_{52} and $1/z$, we have:

$$\begin{aligned} u'_{52} &\sim \frac{4}{v_{52}}, \\ v'_{52} &\sim \frac{2(1-8u_{52})}{u_{52}(2u_{52}-1)}, \\ J_{52} &= -\frac{1}{8}u_{52}(2u_{52}-1)^3v_{52}^2, \\ EJ_{52} &\sim -2, \\ \frac{E'}{E} &\sim -\frac{2+\alpha_1+\alpha_2}{2z} - \frac{1-\alpha_1-\alpha_2}{4u_{52}z}. \end{aligned}$$

We introduce the following coordinate change for convenience in order to make \mathcal{L}_0^* invisible in the chart:

$$\tilde{u}_{52} = \frac{u_{52}}{u_{52} - \frac{1}{2}}.$$

Now, in the (\tilde{u}_{52}, v_{52}) coordinate system, $\mathcal{L}_2^* \setminus \mathcal{L}_0^*$ is given by the equation $v_{52} = 0$ and parametrized by $\tilde{u}_{52} \in \mathbb{C}$, while the line \mathcal{L}_5^* without one point is given by the equation $\tilde{u}_{52} = 0$ and parametrized by $v_{52} \in \mathbb{C}$.

For $v_{52} \rightarrow 0$ and bounded \tilde{u}_{52} and $\frac{1}{z}$, we have:

$$\tilde{u}'_{52} \sim -\frac{8(\tilde{u}_{52}-1)^2}{v_{52}}, \quad (4.4a)$$

$$v'_{52} \sim -\frac{4}{\tilde{u}_{52}} + 8 - 12\tilde{u}_{52}, \quad (4.4b)$$

$$J_{52} = -\frac{1}{16} \frac{\tilde{u}_{52}v_{52}^2}{(\tilde{u}_{52}-1)^4}, \quad (4.4c)$$

$$EJ_{52} \sim -2, \quad (4.4d)$$

$$\frac{E'}{E} \sim -\frac{3}{2z} + \frac{1-\alpha_1-\alpha_2}{2\tilde{u}_{52}z}. \quad (4.4e)$$

We also have:

$$\begin{aligned} \tilde{J}_{52} &= \frac{\partial \tilde{u}_{52}}{\partial u} \frac{\partial v_{52}}{\partial v} - \frac{\partial \tilde{u}_{52}}{\partial v} \frac{\partial v_{52}}{\partial u} = \frac{\tilde{u}_{52}v_{52}^2}{8(1-\tilde{u}_{52})^2}, \\ E\tilde{J}_{52} &= 2(1-\tilde{u}_{52})(2-2\tilde{u}_{52}+\tilde{u}_{52}v_{52}). \end{aligned}$$

From (4.4e) and (4.4b), we get:

$$\frac{E'}{E} \sim -\frac{3}{2z} - \frac{1-\alpha_1-\alpha_2}{8z}v'_{52} + \frac{1-\alpha_1-\alpha_2}{z} - 3\frac{1-\alpha_1-\alpha_2}{2z}\tilde{u}_{52}.$$

Integrating from z_0 to z_1 , we obtain:

$$\begin{aligned} \log \frac{E(z_1)}{E(z_0)} &\sim \log \left(\frac{z_1}{z_0} \right)^{-1/2-\alpha_1-\alpha_2} \\ &\quad - \frac{1-\alpha_1-\alpha_2}{8} \left(\frac{v_{52}(z_1)}{z_1} - \frac{v_{52}(z_0)}{z_0} + \int_{z_0}^{z_1} \frac{v_{52}(z)}{z^2} dz \right. \\ &\quad \left. + 12 \int_{z_0}^{z_1} \frac{\tilde{u}_{52}(z)}{z} dz \right). \end{aligned}$$

Therefore $E(z_1)/E(z_0) \sim 1$, if for all z on the segment from z_0 to z_1 we have $|z-z_0| \ll |z_0|$ and

$|v_{52}(z)| \ll |z_0|$, $|\tilde{u}_{52}(z)| \ll |z_0|$. We choose z_0 on the boundary of D_5 from the proof of Lemma 4.5. Then we have

$$\frac{d(\zeta)}{d(z_0)} \sim E(z_0)d(\zeta) \sim E(z_0)\frac{J_{82}(z_0)}{2} \sim 1 \quad \text{and} \quad |u_{82}(z_0)| = R_5,$$

which implies that

$$|v_{52}| = \left| \frac{1}{u_{82} + \frac{1-\alpha_1-\alpha_2}{8z}} \right| \sim \frac{1}{R_5} \ll 1.$$

Furthermore, equations (4.4c) and (4.4d) imply that:

$$\left| \frac{\tilde{u}_{52}(z_0)}{(\tilde{u}_{52} - 1)^4} \right| = \frac{16|J_{52}(z_0)|}{|v_{52}(z_0)|^2} \sim 8|d(\zeta)|R_5^2,$$

which is small when $|d(\zeta)|$ is sufficiently small.

Since D_5 is an approximate disk with centre ζ and small radius $\sim |d(\zeta)|R_5$, and $R_5 \gg |\zeta|^{-1}$, we have that $|u_{82}(z)| \geq R_5 \gg 1$ hence:

$$|\tilde{u}_{52}(z)| \ll 1 \quad \text{if} \quad z = \zeta + r(z_0 - \zeta), \quad r \geq 1,$$

and

$$\frac{|z - z_0|}{|z_0|} = (r - 1) \left| 1 - \frac{\zeta}{z_0} \right| \ll 1 \quad \text{if} \quad r - 1 \ll \frac{1}{|1 - \frac{\zeta}{z_0}|}.$$

Then equations (4.4c), (4.4d) and $E \sim d(\zeta)^{-1}$ yield

$$v_{52}^{-1} \sim \left(-\frac{\tilde{u}_{52}}{d(\zeta)} \right)^{1/2},$$

which in combination with (4.2b) leads to

$$\frac{d\tilde{u}_{52}^{1/2}}{dz} \sim (-d(\zeta))^{-1/2},$$

hence

$$\tilde{u}_{52}^{1/2} \sim \tilde{u}_{52}(z_0)^{1/2} + (-d(\zeta))^{-1/2}(z - z_0),$$

and therefore

$$\tilde{u}_{52}(z) \sim -\frac{(z - z_0)^2}{d(\zeta)} \quad \text{if} \quad |z - z_0| \gg |\tilde{u}_{52}(z_0)|^{1/2}.$$

For large finite $R_2 > 0$, the equation $|\tilde{u}_{52}| = R_2$ corresponds to $|z - z_0| \sim \sqrt{|d(\zeta)R_2|}$, which is still small compared to $|z_0| \sim |\zeta|$, and therefore $|z - \zeta| \leq |z - z_0| + |z_0 - \zeta| \ll |\zeta|$. This proves the statement of the lemma. \square

LEMMA 4.7 Behaviour near $\mathcal{L}_6^* \setminus \mathcal{L}_3^*$. *If a solution at the complex time z is sufficiently close to $\mathcal{L}_6^* \setminus \mathcal{L}_3^*$, then there exists a unique $\zeta \in \mathbb{C}$ such that:*

- (i) $v_{91}(\zeta) = 0$, i.e. $v_{91}(\zeta) \in \mathcal{L}_9(\zeta)$;
- (ii) $|z - \zeta| = O(|d(z)||v_{91}(z)|)$ for small $d(z)$ and limited $|v_{91}(z)|$.

In other words, the solution has a pole at $z = \zeta$.

For large $R_6 > 0$, consider the set $\{z \in \mathbb{C} \mid |v_{91}| \leq R_6\}$. Then, its connected component containing ζ is an approximate disk D_6 with centre ζ and radius $|d(\zeta)|R_6$, and $z \mapsto v_{91}(z)$ is a complex analytic diffeomorphism from that approximate disk onto $\{v \in \mathbb{C} \mid |v| \leq R_6\}$.

Proof. Line $\mathcal{L}_6^* \setminus \mathcal{L}_3^*$ is given by the equation $u_{91} = 0$ and parametrized by $v_{91} \in \mathbb{C}$, see Section A.4. Moreover, \mathcal{L}_9 (without one point), is given by $v_{91} = 0$ and parametrized by $u_{91} \in \mathbb{C}$. For the study of the solutions near $\mathcal{L}_6^* \setminus \mathcal{L}_3^*$, we use the coordinates (u_{91}, v_{91}) . Asymptotically, for $u_{91} \rightarrow 0$, and bounded v_{91} , $1/z$, we have:

$$\begin{aligned} v'_{91} &\sim -\frac{1}{u_{91}}, \\ J_{91} &= u_{91}, \\ \frac{J'_{91}}{J_{91}} &= -2 + \frac{3}{2z} + O(u_{91}) = -2 + \frac{3}{2z} + O(J_{91}), \\ EJ_{91} &\sim -1 + \frac{\alpha_1}{zv_{91}}. \end{aligned}$$

Notice that these equations are analogous to (4.1a)-(4.1d), thus the remainder of the proof is similar to that provided for Lemma 4.3. \square

LEMMA 4.8 Behaviour near $\mathcal{L}_3^* \setminus \mathcal{L}_0^*$. For large finite $R_3 > 0$, consider the set of all $z \in \mathbb{C}$, such that the solution at complex time z is close to $\mathcal{L}_3^* \setminus \mathcal{L}_0^*$, with $|v_{61}(z)| \leq R_1$, but not close to \mathcal{L}_6^* . Then the connected component of that set containing ζ is the complement of D_6 in an approximate disk D_3 with centre at ζ and radius $\sim \sqrt{|d(\zeta)|R_3}$. More precisely, $z \mapsto v_{61}$ defines a 2-fold covering from the annular domain $D_3 \setminus D_6$ onto the complement in $\{u \in \mathbb{C} \mid |u| \leq R_3\}$ of an approximate disk with centre at the origin and small radius $\sim |d(\zeta)|R_6^2$, where $v_{61}(z) \sim -d(\zeta)(z - \zeta)^2$.

Proof. The line $\mathcal{L}_3^* \setminus \mathcal{L}_0^*$ is visible in the coordinate system (u_{61}, v_{61}) , where it is given by the equation $u_{61} = 0$ and parametrized by $v_{61} \in \mathbb{C}$; see Section A.3. In that chart, the line \mathcal{L}_6^* (without one point) is given by the equation $v_{61} = 0$ and parametrized by $u_{61} \in \mathbb{C}$.

For $u_{61} \rightarrow 0$ and bounded v_{61} and $1/z$, we have:

$$\begin{aligned} u'_{61} &\sim \frac{1}{v_{61}}, \\ v'_{61} &\sim -\frac{2}{u_{61}}, \\ J_{61} &= u_{61}^2 v_{61}, \\ EJ_{61} &\sim -1, \\ \frac{E'}{E} &\sim -\frac{3}{2z} - \frac{\alpha_1}{v_{61}z}. \end{aligned}$$

Notice that these equations are analogous to (4.2a)-(4.2e). Therefore, the remainder of the proof is similar to that provided for Lemma 4.4. \square

THEOREM 4.9. Let $\varepsilon_1, \varepsilon_2, \varepsilon_3$ be given such that $\varepsilon_1 > 0$, $0 < \varepsilon_2 < \frac{3}{2}$, $0 < \varepsilon_3 < 1$. Then there exists $\delta > 0$ such that if $|z_0| > \varepsilon_1$ and $|d(z_0)| < \delta$, then:

$$\rho = \sup\{r > |z_0| \text{ such that } |d(z)| < \delta \text{ whenever } |z_0| \leq |z| \leq r\}$$

satisfies:

- (i) $\delta \geq |d(z_0)| \left(\frac{\rho}{|z_0|} \right)^{3/2-\varepsilon_2} (1 - \varepsilon_3)$;
- (ii) if $|z_0| \leq |z| \leq \rho$ then $d(z) = d(z_0) \left(\frac{z}{z_0} \right)^{3/2+\varepsilon_2(z)} (1 + \varepsilon_3(z))$;

(iii) if $|z| \geq \rho$ then $d(z) \geq \delta(1 - \varepsilon_3)$.

Proof. Suppose a solution of the system (2.3) is close to \mathcal{L}_0^* at times z_0 and z_1 . It follows from Lemmas 4.3–4.8 that for every solution close to \mathcal{I} , the set of complex times z such that the solution is not close to \mathcal{L}_0^* is the union of approximate disks of radius $\sim |d|^{1/2}$. Hence if the solution is near \mathcal{I} for all complex times z such that $|z_0| \leq |z| \leq |z_1|$, then there exists a path γ from z_0 to z_1 , such that the solution is close to \mathcal{L}_0 for all $z \in \gamma$ and γ is C^1 -close to the path: $t \mapsto z_1^t z_0^{1-t}$, $t \in [0, 1]$,

Then Lemma 4.1 implies that:

$$\log \frac{E(z)}{E(z_0)} = -\frac{3}{2} \log \frac{z}{z_0} \int_0^1 dt + o(1),$$

therefore

$$E(z) = E(z_0) \left(\frac{z}{z_0} \right)^{3/2+o(1)} (1 + o(1)),$$

and

$$d(z) = d(z_0) \left(\frac{z}{z_0} \right)^{3/2+o(1)} (1 + o(1)). \quad (4.5)$$

From Lemmas 4.3–4.8 we then have that, as long as the solution is close to \mathcal{I} , as it moves into a neighbourhood of $\mathcal{L}_4^* \setminus \mathcal{L}_1^*$, $\mathcal{L}_5^* \setminus \mathcal{L}_2^*$, $\mathcal{L}_6^* \setminus \mathcal{L}_3^*$, the ratio of d remains close to 1.

For the first statement of the theorem, we have:

$$\delta > d(z) \geq d(z_0) \left(\frac{z}{z_0} \right)^{3/2-\varepsilon_2} (1 - \varepsilon_3)$$

and so

$$\delta \geq \sup_{\{z \mid |d(z)| < \delta\}} d(z_0) \left(\frac{z}{z_0} \right)^{3/2-\varepsilon_2} (1 - \varepsilon_3).$$

The second statement follows from (4.5), while the third follows by the assumption on z . \square

5. The limit set

In this section we define and consider properties of the limit sets of solutions. In Theorem 5.1, we prove that there is a compact set $K \subset \mathcal{F}(\infty)$, such the limit sets of all solutions of (2.1) are contained in K and that the limit set of any solution is non-empty, compact, connected, and invariant under the flow of the autonomous system (2.4). These results lead us to Theorem 5.4, i.e., that each non-rational solution of the fourth Painlevé equation has infinitely many zeroes and poles.

THEOREM 5.1. *There exists a compact subset K of $\mathcal{F}(\infty) \setminus \mathcal{I}(\infty)$, such that the limit set $\Omega_{(u,v)}$ of any solution (u, v) is contained in K . Moreover, $\Omega_{(u,v)}$ is a non-empty, compact and connected set, which is invariant under the flow of the autonomous system (2.4).*

Proof. For any positive numbers δ, r , let $K_{\delta,r}$ denote the set of all $s \in \mathcal{F}(z)$ such that $|z| \geq r$ and $|d(s)| \geq \delta$. Since $\mathcal{F}(z)$ is a complex analytic family over $\mathbb{P}^1 \setminus \{0\}$ of compact surfaces $\mathcal{F}(z)$, $K_{\delta,r}$ is also compact. Furthermore $K_{\delta,r}$ is disjoint from the union of the infinity sets $\mathcal{I}(z)$, $z \in \mathbb{P}^1 \setminus \{0\}$, and therefore $K_{\delta,r}$ is a compact subset of Okamoto's space $\mathcal{O} \setminus \mathcal{F}(\infty)$. When r grows to the

infinity, the sets $K_{\delta,r}$ shrink to the set

$$K_{\delta,\infty} = \{s \in \mathcal{F}(\infty) \mid |d(s)| \geq \delta\} \subset \mathcal{F}(\infty) \setminus \mathcal{I}(\infty),$$

which is compact.

It follows from Theorem 4.9 that there exists $\delta > 0$ such that for every solution (u, v) there exists $r_0 > 0$ with the following property:

$$(u(z), v(z)) \in K_{\delta,r_0} \text{ for every } z \text{ such that } |z| \geq r_0.$$

In the sequel, we take $r \geq r_0$, when it follows that $(u(z), v(z)) \in K_{\delta,r}$ whenever $|z| \geq r$. Let $Z_r = \{z \in \mathbb{C} \mid |z| \geq r\}$ and let $\Omega_{(u,v),r}$ denote the closure of $(u, v)(Z_r)$ in \mathcal{O} . Since Z_r is connected and (u, v) is continuous, $\Omega_{(u,v),r}$ is also connected. Since $(u, v)(Z_r)$ is contained in the compact subset $K_{\delta,r}$, its closure $\Omega_{(u,v),r}$ is also contained in $K_{\delta,r}$ and therefore $\Omega_{(u,v),r}$ is a non-empty compact and connected subset of $\mathcal{O} \setminus \mathcal{F}(\infty)$. The intersection of a decreasing sequence of non-empty compact and connected sets is non-empty, compact, and connected: therefore, as $\Omega_{(u,v),r}$ decrease to $\Omega_{(u,v)}$ when r grows to the infinity, it follows that $\Omega_{(u,v)}$ is a non-empty, compact, and connected set of \mathcal{O} . Since $\Omega_{(u,v),r} \subset K_{\delta,r}$ for all $r \geq r_0$, and the sets $K_{\delta,r}$ shrink to the compact subset $K_{\delta,\infty}$ of $\mathcal{F}(\infty) \setminus \mathcal{I}(\infty)$ as r grows to the infinity, it follows that $\Omega_{(u,v)} \subset K_{\delta,\infty}$. This proves the first statement of the theorem with $K = K_{\delta,\infty}$.

Since $\Omega_{(u,v)}$ is the intersection of the decreasing family of compact sets $\Omega_{(u,v),r}$, there exists for every neighbourhood A of $\Omega_{(u,v)}$ in \mathcal{O} and $r > 0$ such that $\Omega_{(u,v),r} \subset A$, hence $(u(z), v(z)) \in A$ for every $z \in \mathbb{C}$ such that $|z| \geq r$. If z_j is any sequence in $\mathbb{C} \setminus \{0\}$ such that $|z_j| \rightarrow \infty$, then the compactness of $K_{\delta,r}$, in combination with $(u, v)Z_r \subset K_{\delta,r}$, implies that there is a subsequence $j = j(k) \rightarrow \infty$ as $k \rightarrow \infty$ and an $s \in K_{\delta,r}$, such that:

$$(u(z_{j(k)}), v(z_{j(k)})) \rightarrow s \text{ as } k \rightarrow \infty.$$

Then it follows that $s \in \Omega_{(u,v)}$.

Next, we prove that $\Omega_{(u,v)}$ is invariant under the flow Φ^t of the autonomous Hamiltonian system. Let $s \in \Omega_{(u,v)}$ and z_j be a sequence in $\mathbb{C} \setminus \{0\}$ such that $z_j \rightarrow \infty$ and $(u(z_j), v(z_j)) \rightarrow s$. Since the z -dependent vector field of the Butroux-Painlevé system converges in C^1 to the vector field of the autonomous Hamiltonian system as $z \rightarrow \infty$, it follows from the continuous dependence on initial data and parameters, that the distance between $(u(z_j + t), v(z_j + t))$ and $\Phi^t(u(z_j), v(z_j))$ converges to zero as $j \rightarrow \infty$. Since $\Phi^t(u(z_j), v(z_j)) \rightarrow \Phi^t(s)$ and $z_j \rightarrow \infty$ as $j \rightarrow \infty$, it follows that $(u(z_j + t), v(z_j + t)) \rightarrow \Phi^t(s)$ and $z_j + t \rightarrow \infty$ as $j \rightarrow \infty$, hence $\Phi^t(s) \in \Omega_{(u,v)}$. \square

PROPOSITION 5.2. *Every non-special solution $(u(z), v(z))$ intersects each of the pole lines $\mathcal{L}_7, \mathcal{L}_8, \mathcal{L}_9$ infinitely many times.*

Proof. First, suppose that a solution $(u(z), v(z))$ intersects the union $\mathcal{L}_7 \cup \mathcal{L}_8 \cup \mathcal{L}_9$ only finitely many times.

According to Theorem 5.1, the limit set $\Omega_{(u,v)}$ is a compact set in $\mathcal{F}(\infty) \setminus \mathcal{I}(\infty)$. If $\Omega_{(u,v)}$ intersects one of the three pole lines $\mathcal{L}_7, \mathcal{L}_8, \mathcal{L}_9$ at a point p , then there exists arbitrarily large z such that $(u(z), v(z))$ is arbitrarily close to p , when the transversality of the vector field to the pole line implies that $(u(\zeta), v(\zeta)) \in \mathcal{L}_7 \cup \mathcal{L}_8 \cup \mathcal{L}_9$ for a unique ζ near z . As this would imply that $(u(z), v(z))$ intersects $\mathcal{L}_7 \cup \mathcal{L}_8 \cup \mathcal{L}_9$ has infinitely many times, it follows that $\Omega_{(u,v)}$ is a compact subset of $\mathcal{F}(\infty) \setminus (\mathcal{I}(\infty) \cup \mathcal{L}_7(\infty) \cup \mathcal{L}_8(\infty) \cup \mathcal{L}_9(\infty))$. However, $\mathcal{L}_7(\infty) \cup \mathcal{L}_8(\infty) \cup \mathcal{L}_9(\infty)$ is equal to the set of all points in $\mathcal{F}(\infty) \setminus \mathcal{I}(\infty)$ which project to the line \mathcal{L}_0 , and therefore $\mathcal{F}(\infty) \setminus (\mathcal{I}(\infty) \cup \mathcal{L}_7(\infty) \cup \mathcal{L}_8(\infty) \cup \mathcal{L}_9(\infty))$ is the affine (u, v) coordinate chart, of which $\Omega_{(u,v)}$ is a compact subset, which implies that $u(z)$ and $v(z)$ remain bounded for large $|z|$. It follows

from boundedness of u and v that $u(z)$ and $v(z)$ are equal to holomorphic functions of $1/z$ in a neighbourhood of $z = \infty$, which implies that there are complex numbers $u(\infty)$, $v(\infty)$ which are the limit points of $u(z)$ and $v(z)$ as $|z| \rightarrow \infty$. In other words, $\Omega_{(u,v)} = \{(u(\infty), v(\infty))\}$ is a one point set. That means that the solution is analytic at infinity, i.e., it is analytic on the whole \mathbb{CP}^1 , thus it must be rational.

Since the limit set $\Omega_{(u,v)}$ is invariant under the autonomous flow, it means that it will contain the whole irreducible component of a cubic curve: $-uv(u + v + 2) = c$, for some constant c . As shown in Section 3, such a curve is reducible for $c = 0$, and the special solutions correspond to each of the irreducible components. In all other cases, all three base points b_0, b_1, b_2 on the line \mathcal{L}_0 will be contained in the limit set, which are projections of the pole lines $\mathcal{L}_7(\infty)$, $\mathcal{L}_8(\infty)$, $\mathcal{L}_9(\infty)$ respectively. Thus, a non-special solution will intersect each of them infinitely many times. \square

REMARK 5.3. *The limit set $\Omega_{(u,v)}$ is invariant under the autonomous Hamiltonian system. If it contains only one point, as we obtained in the proof of Theorem 5.2, that point must be an equilibrium point of the autonomous Hamiltonian system (2.4), that is:*

$$(u(\infty), v(\infty)) \in \left\{ (0, 0), (0, -2), (-2, 0), \left(-\frac{2}{3}, -\frac{2}{3} \right) \right\}.$$

These are limiting values of the rational solutions, see Section 3.2.

THEOREM 5.4. *Every non-special solution of the fourth Painlevé equation (1.1) has infinitely many poles and infinitely many zeros.*

Proof. It is enough to prove that a non-special solution u of (2.3) has infinitely many poles and zeroes. Notice that at the intersection point with \mathcal{L}_7 , u has a pole and v a zero; at the intersection with \mathcal{L}_8 both have poles, and on \mathcal{L}_9 , u has a zero and v a pole. Since it is shown in Proposition 5.2 that (u, v) intersects each of the lines \mathcal{L}_7 , \mathcal{L}_8 , \mathcal{L}_9 infinitely many times, the statement is proved. \square

A. Resolution of the Painlevé vector field

A.1 The affine charts

A.1.1 *Affine Chart* (u_{01}, v_{01}) The first affine chart is defined by the original coordinates

$$\begin{aligned} u_{01} &= u, \\ v_{01} &= v, \\ E &= -uv(u + v + 2). \end{aligned}$$

A.1.2 *Affine Chart* (u_{02}, v_{02}) The second affine chart is given by the following coordinates:

$$\begin{aligned} u_{02} &= \frac{1}{u}, & v_{02} &= \frac{v}{u}, \\ u &= \frac{1}{u_{02}}, & v &= \frac{v_{02}}{u_{02}}. \end{aligned}$$

The line at the infinity is $\mathcal{L}_0 : u_{02} = 0$.

The Painlevé vector field is given by

$$\begin{aligned} u'_{02} &= 1 + 2u_{02} + 2v_{02} + \frac{1}{2z}(2\alpha_1 u_{02}^2 + u_{02}), \\ v'_{02} &= \frac{v_{02}}{u_{02}}(4u_{02} + 3v_{02} + 3) + \frac{1}{z}(-\alpha_2 u_{02} + \alpha_1 u_{02} v_{02}), \end{aligned}$$

which contain base points at

$$b_0 : u_{02} = 0, v_{02} = 0 \quad \text{and} \quad b_1 : u_{02} = 0, v_{02} = -1.$$

The energy is

$$\begin{aligned} E &= -\frac{v_{02}(1 + 2u_{02} + v_{02})}{u_{02}^3}, \\ E' &= \frac{1}{2u_{02}^3 z} (4\alpha_1 u_{02}^2 v_{02} + 2\alpha_1 u_{02} v_{02}^2 + 4\alpha_2 u_{02}^2 + 3v_{02}^2 \\ &\quad + 4(\alpha_1 + \alpha_2 + 1)u_{02} v_{02} + 2\alpha_2 u_{02} + 3v_{02}). \end{aligned}$$

A.1.3 Affine Chart (u_{03}, v_{03}) We have the coordinates

$$\begin{aligned} u_{03} &= \frac{1}{v}, & v_{03} &= \frac{u}{v}, \\ u &= \frac{v_{03}}{u_{03}}, & v &= \frac{1}{u_{03}}, \end{aligned}$$

and the line at the infinity is given by $\mathcal{L}_0 : u_{03} = 0$.

The flow is given by

$$\begin{aligned} u'_{03} &= -1 - 2u_{03} - 2v_{03} + \frac{1}{2z}(2\alpha_2 u_{03}^2 + u_{03}), \\ v'_{03} &= -\frac{v_{03}}{u_{03}}(4u_{03} + 3v_{03} + 3) + \frac{1}{z}(-\alpha_1 u_{03} + \alpha_2 u_{03} v_{03}), \end{aligned}$$

which contains a base point at

$$b_2 : u_{03} = 0, v_{03} = 0,$$

and $(u_{03} = 0, v_{03} = -1)$, which is b_1 .

The energy is given by

$$\begin{aligned} E &= -\frac{v_{03}(1 + 2u_{03} + v_{03})}{u_{03}^3}, \\ E' &= \frac{1}{2u_{03}^3 z} (4\alpha_2 u_{03}^2 v_{03} + 2\alpha_2 u_{03} v_{03}^2 + 4\alpha_1 u_{03}^2 + 3v_{03}^2 \\ &\quad + 4(\alpha_1 + \alpha_2 + 1)u_{03} v_{03} + 2\alpha_1 u_{03} + 3v_{03}). \end{aligned}$$

A.2 Resolution at base points b_0, b_1, b_2

A.2.1 Resolution at b_0 The first chart is given by the coordinate change:

$$\begin{aligned} u_{11} &= \frac{u_{02}}{v_{02}} = \frac{1}{v}, & v_{11} &= v_{02} = \frac{v}{u}, \\ u &= \frac{1}{u_{11} v_{11}}, & v &= \frac{1}{u_{11}}. \end{aligned}$$

The exceptional line is $\mathcal{L}_1 : v_{11} = 0$. The preimage of line \mathcal{L}_0 is visible in this chart, and given by the equation $u_{11} = 0$.

The flow in this chart:

$$\begin{aligned} u'_{11} &= -\frac{1}{v_{11}}(v_{11} + 2u_{11}v_{11} + 2) + \frac{u_{11}}{2z}(1 + 2\alpha_2 u_{11}), \\ v'_{11} &= \frac{1}{u_{11}}(3v_{11} + 4u_{11}v_{11} + 3) + \frac{u_{11}v_{11}}{z}(-\alpha_2 + \alpha_1 v_{11}), \end{aligned}$$

contains no new base points.

The energy is given by

$$\begin{aligned} E &= -\frac{1 + v_{11} + 2u_{11}v_{11}}{u_{11}^3 v_{11}^2}, \\ E' &= \frac{1}{2u_{11}^3 v_{11}^2 z} (3 + 2\alpha_2 u_{11} + 3v_{11} + 4(1 + \alpha_1 + \alpha_2)u_{11}v_{11} \\ &\quad + 4\alpha_2 u_{11}^2 v_{11} + 2\alpha_1 u_{11}v_{11}^2 + 4\alpha_1 u_{11}^2 v_{11}^2). \end{aligned}$$

The second chart is given by

$$\begin{aligned} u_{12} &= u_{02} = \frac{1}{u}, & v_{12} &= \frac{v_{02}}{u_{02}} = v, \\ u &= \frac{1}{u_{12}}, & v &= v_{12}. \end{aligned}$$

The exceptional line is $\mathcal{L}_1 : u_{12} = 0$. The preimage of line \mathcal{L}_0 is not visible in this chart.

The flow is

$$\begin{aligned} u'_{12} &= 1 + 2u_{12} + 2u_{12}v_{12} + \frac{u_{12}}{2z}(1 + 2\alpha_1 u_{12}), \\ v'_{12} &= \frac{v_{12}}{u_{12}}(2u_{12} + 2 + u_{12}v_{12}) - \frac{1}{2z}(2\alpha_2 + v_{12}). \end{aligned}$$

Both the vector field and the anticanonical pencil have base point at

$$b_3 : u_{12} = 0, v_{12} = 0.$$

The energy is given by

$$\begin{aligned} E &= -\frac{v_{12}(1 + 2u_{12} + u_{12}v_{12})}{u_{12}^2}, \\ E' &= \frac{1}{2u_{12}^2 z} (2\alpha_2 + 4\alpha_2 u_{12} + 3v_{12} + 4(1 + \alpha_1 + \alpha_2)u_{12}v_{12} \\ &\quad + 4\alpha_1 u_{12}^2 v_{12} + 3u_{12}v_{12}^2 + 2\alpha_1 u_{12}^2 v_{12}^2). \end{aligned}$$

A.2.2 Resolution at b_1 The first chart is given by the coordinate change:

$$\begin{aligned} u_{21} &= \frac{u_{02}}{v_{02} + 1} = \frac{1}{u + v}, & v_{21} &= v_{02} + 1 = \frac{u + v}{u}, \\ u &= \frac{1}{u_{21}v_{21}}, & v &= \frac{v_{21} - 1}{u_{21}v_{21}}. \end{aligned}$$

The exceptional line is $\mathcal{L}_2 : v_{21} = 0$. The preimage of the line \mathcal{L}_0 is visible in this chart, and given by the equation $u_{21} = 0$.

The flow is given by

$$\begin{aligned} u'_{21} &= \frac{(2u_{21} + 1)(2 - v_{21})}{v_{21}} + \frac{u_{21}}{2z}(2(\alpha_1 + \alpha_2)u_{21} + 1), \\ v'_{21} &= \frac{(4u_{21} + 3)(v_{21} - 1)}{u_{21}} + \frac{u_{21}v_{21}}{z}(\alpha_1 v_{21} - \alpha_1 - \alpha_2), \end{aligned}$$

and contains a new base point at

$$b_4 : u_{21} = -\frac{1}{2}, v_{21} = 0.$$

The energy is given by

$$\begin{aligned} E &= -\frac{(2u_{21} + 1)(v_{21} - 1)}{u_{21}^3 v_{21}^2}, \\ E' &= \frac{1}{2u_{21}^3 v_{21}^2 z}(-3 - 2(2 + \alpha_1 + \alpha_2)u_{21} + 3v_{21} + 4(1 + \alpha_2)u_{21}v_{21} \\ &\quad + 4(\alpha_2 - \alpha_1)u_{21}^2 v_{21} + 2\alpha_1 u_{21}v_{21}^2 + 4\alpha_1 u_{21}^2 v_{21}^2). \end{aligned}$$

The second chart is given by

$$\begin{aligned} u_{22} = u_{02} &= \frac{1}{u}, & v_{22} &= \frac{v_{02} + 1}{u_{02}} = u + v, \\ u &= \frac{1}{u_{22}}, & v &= v_{22} - \frac{1}{u_{22}}. \end{aligned}$$

The exceptional line is $\mathcal{L}_2 : u_{22} = 0$. The preimage of line \mathcal{L}_0 is not visible in this chart.

The flow is given by

$$\begin{aligned} u'_{22} &= -1 + 2u_{22} + 2u_{22}v_{22} + \frac{u_{22}}{2z}(1 + 2\alpha_1 u_{22}), \\ v'_{22} &= \frac{(v_{22} + 2)(u_{22}v_{22} - 2)}{u_{22}} - \frac{1}{2z}(v_{22} + 2\alpha_1 + 2\alpha_2), \end{aligned}$$

and contains a base point $(u_{22} = 0, v_{22} = -2)$, which is b_4 .

The energy is given by

$$\begin{aligned} E &= \frac{(2 + v_{22})(1 - u_{22}v_{22})}{u_{22}^2}, \\ E' &= \frac{1}{2u_{22}^2 z}(-2(2 + \alpha_1 + \alpha_2) + 4(\alpha_2 - \alpha_1)u_{22} - 3v_{22} + 4(1 + \alpha_2)u_{22}v_{22} \\ &\quad + 4\alpha_1 u_{22}^2 v_{22} + 3u_{22}v_{22}^2 + 2\alpha_1 u_{22}^2 v_{22}^2). \end{aligned}$$

A.2.3 Resolution at b_2 The first chart is given by

$$\begin{aligned} u_{31} = \frac{u_{03}}{v_{03}} &= \frac{1}{u}, & v_{31} = v_{03} &= \frac{u}{v}, \\ u &= \frac{1}{u_{31}}, & v &= \frac{1}{u_{31}v_{31}}. \end{aligned}$$

The exceptional line is $\mathcal{L}_3 : v_{31} = 0$. The preimage of line \mathcal{L}_0 is visible in this chart, and given by the equation $u_{31} = 0$.

The flow

$$\begin{aligned} u'_{31} &= \frac{v_{31} + 2u_{31}v_{31} + 2}{v_{31}} + \frac{u_{31}}{2z}(1 + 2\alpha_1 u_{31}), \\ v'_{31} &= -\frac{3 + 4u_{31}v_{31} + 3v_{31}}{u_{31}} + \frac{u_{31}v_{31}}{z}(-\alpha_1 + \alpha_2 v_{31}), \end{aligned}$$

contains no base point.

The energy is given by

$$\begin{aligned} E &= -\frac{1 + v_{31} + 2u_{31}v_{31}}{u_{31}^3 v_{31}^2}, \\ E' &= \frac{1}{2u_{31}^3 v_{31}^2 z} (3 + 2\alpha_1 u_{31} + 3v_{31} + 4(1 + \alpha_1 + \alpha_2)u_{31}v_{31} \\ &\quad + 4\alpha_1 u_{31}^2 v_{31} + 2\alpha_2 u_{31}v_{31}^2 + 4\alpha_2 u_{31}^2 v_{31}^2). \end{aligned}$$

The second chart is given by

$$\begin{aligned} u_{32} &= u_{03} = \frac{1}{v}, & v_{32} &= \frac{v_{03}}{u_{03}} = u, \\ u &= v_{32}, & v &= \frac{1}{u_{32}}. \end{aligned}$$

The exceptional line is $\mathcal{L}_3 : u_{32} = 0$. The preimage of line \mathcal{L}_0 is not visible in this chart.

The flow is

$$\begin{aligned} u'_{32} &= -1 - 2u_{32} - 2u_{32}v_{32} + \frac{u_{32}}{2z}(1 + 2\alpha_2 u_{32}), \\ v'_{32} &= -\frac{v_{32}}{u_{32}}(2u_{32} + 2 + v_{32}u_{32}) - \frac{1}{2z}(2\alpha_1 + v_{32}). \end{aligned}$$

Both the vector field and the anticanonical pencil have a base point at

$$b_5 : u_{32} = 0, v_{32} = 0.$$

The energy is given by

$$\begin{aligned} E &= -\frac{v_{32}(1 + 2u_{32} + u_{32}v_{32})}{u_{32}^2}, \\ E' &= \frac{1}{2u_{32}^2 z} (2\alpha_1 + 4\alpha_1 u_{32} + 3v_{32} + 4(1 + \alpha_1 + \alpha_2)u_{32}v_{32} \\ &\quad + 4\alpha_2 u_{32}^2 v_{32} + 3u_{32}v_{32}^2 + 2\alpha_2 u_{32}^2 v_{32}^2). \end{aligned}$$

A.3 Resolution at points b_3, b_4, b_5

A.3.1 *Resolution at b_3* The first chart is

$$\begin{aligned} u_{41} &= \frac{u_{12}}{v_{12}} = \frac{1}{uv}, & v_{41} &= v_{12} = v, \\ u &= \frac{1}{u_{41}v_{41}}, & v &= v_{41}, \end{aligned}$$

and the corresponding Jacobian is

$$J_{41} = \frac{\partial u_{41}}{\partial u} \frac{\partial v_{41}}{\partial v} - \frac{\partial u_{41}}{\partial v} \frac{\partial v_{41}}{\partial u} = -\frac{1}{u^2 v} = -u_{41}^2 v_{41}.$$

The exceptional line is $\mathcal{L}_4 : v_{41} = 0$. The preimage of line \mathcal{L}_1 in this chart is $u_{41} = 0$. \mathcal{L}_0 is not visible in this chart.

The flow is given by

$$\begin{aligned} u'_{41} &= -\frac{1}{v_{41}} + u_{41}v_{41} + \frac{u_{41}(\alpha_2 + v_{41} + \alpha_1 u_{41}v_{41}^2)}{zv_{41}}, \\ v'_{41} &= \frac{2}{u_{41}} + 2v_{41} + v_{41}^2 - \frac{1}{2z}(2\alpha_2 + v_{41}), \end{aligned}$$

and contains a base point:

$$b_6 : u_{41} = \frac{z}{\alpha_2}, v_{41} = 0.$$

The energy and related quantities are

$$\begin{aligned} E &= -\frac{1 + 2u_{41}v_{41} + u_{41}v_{41}^2}{u_{41}^2v_{41}}, \quad EJ_{41} = 1 + 2u_{41}v_{41} + u_{41}v_{41}^2, \\ E' &= \frac{1}{2u_{41}^2v_{41}^2z} (2\alpha_2 + 3v_{41} + 4\alpha_2u_{41}v_{41} + 4(1 + \alpha_1 + \alpha_2)u_{41}v_{41}^2 \\ &\quad + 3u_{41}v_{41}^3 + 4\alpha_1u_{41}^2v_{41}^3 + 2\alpha_1u_{41}^2v_{41}^4). \end{aligned}$$

The second chart is given by

$$\begin{aligned} u_{42} &= u_{12} = \frac{1}{u}, \quad v_{42} = \frac{v_{12}}{u_{12}} = uv, \\ u &= \frac{1}{u_{42}}, \quad v = u_{42}v_{42}. \end{aligned}$$

The exceptional line is $\mathcal{L}_4 : u_{42} = 0$. The preimages of \mathcal{L}_0 and \mathcal{L}_1 are not visible in this chart.

The flow is

$$\begin{aligned} u'_{42} &= 1 + 2u_{42} + 2u_{42}^2v_{42} + \frac{u_{42}}{2z}(1 + 2\alpha_1u_{42}), \\ v'_{42} &= \frac{v_{42}}{u_{42}} - u_{42}v_{42}^2 - \frac{1}{zu_{42}}(\alpha_2 + u_{42}v_{42} + \alpha_1u_{42}^2v_{42}), \end{aligned}$$

and contains a base point $(u_{42} = 0, v_{42} = \frac{\alpha_2}{z})$, which is b_6 .

A.3.2 Resolution at b_4 The first chart is given by

$$\begin{aligned} u_{51} &= \frac{u_{21} + \frac{1}{2}}{v_{21}} = \frac{u(u + v + 2)}{2(u + v)^2}, \quad v_{51} = v_{21} = \frac{u + v}{u}, \\ u &= \frac{2}{v_{51}(2u_{51}v_{51} - 1)}, \quad v = \frac{2(v_{51} - 1)}{v_{51}(2u_{51}v_{51} - 1)}. \end{aligned}$$

The exceptional line is $\mathcal{L}_5 : v_{51} = 0$. The preimage of \mathcal{L}_2 is not visible in this chart, while the preimage of \mathcal{L}_0 is given by $u_{51}v_{51} = \frac{1}{2}$.

The flow is given by

$$\begin{aligned} u'_{51} &= -\frac{2u_{51}(1 + 2u_{51}v_{51}(3v_{51} - 4))}{v_{51}(2u_{51}v_{51} - 1)} + \\ &\quad + \frac{(2u_{51}v_{51} - 1)(\alpha_1 + \alpha_2 - 1 - 4(\alpha_1 + \alpha_2)u_{51}v_{51} + 2\alpha_1u_{51}v_{51}^2)}{4zv_{51}}, \\ v'_{51} &= \frac{2(v_{51} - 1)(4u_{51}v_{51} + 1)}{2u_{51}v_{51} - 1} + \frac{v_{51}(\alpha_1v_{51} - \alpha_1 - \alpha_2)(2u_{51}v_{51} - 1)}{2z}, \end{aligned}$$

and contains a base point

$$b_7 : u_{51} = \frac{1 - \alpha_1 - \alpha_2}{8z}, v_{51} = 0.$$

The second chart is

$$\begin{aligned} u_{52} &= u_{21} + \frac{1}{2} = \frac{1}{u+v} + \frac{1}{2}, & v_{52} &= \frac{v_{21}}{u_{21} + \frac{1}{2}} = \frac{2(u+v)^2}{u(u+v+2)}, \\ u &= \frac{2}{(2u_{52}-1)u_{52}v_{52}}, & v &= \frac{2(u_{52}v_{52}-1)}{(2u_{52}-1)u_{52}v_{52}} \end{aligned}$$

which has Jacobian

$$J_{52} = \frac{\partial u_{52}}{\partial u} \frac{\partial v_{52}}{\partial v} - \frac{\partial u_{52}}{\partial v} \frac{\partial v_{52}}{\partial u} = -\frac{2}{u^2(u+v+2)} = -\frac{1}{8}u_{52}(2u_{52}-1)^3v_{52}^2.$$

The exceptional line is $\mathcal{L}_5 : u_{52} = 0$. In this chart, the preimage of \mathcal{L}_2 is given by $v_{52} = 0$, and of \mathcal{L}_0 by $u_{52} = \frac{1}{2}$.

The flow is given by

$$\begin{aligned} u'_{52} &= -2u_{52} + \frac{4}{v_{52}} + \frac{(2u_{52}-1)(1+(\alpha_1+\alpha_2)(2u_{52}-1))}{4z}, \\ v'_{52} &= \frac{2(1-8u_{52}+6u_{52}^2v_{52})}{u_{52}(2u_{52}-1)} \\ &\quad + \frac{v_{52}(2u_{52}-1)(\alpha_1+\alpha_2-1-4(\alpha_1+\alpha_2)u_{52}+2\alpha_1u_{52}^2v_{52})}{4zu_{52}}, \end{aligned}$$

which contains a base point $(u_{52} = 0, v_{52} = 8z/(1-\alpha_1-\alpha_2))$, which is b_7 .

The energy and related quantities are

$$\begin{aligned} E &= -\frac{16(u_{52}v_{52}-1)}{u_{52}(2u_{52}-1)^3v_{52}^2}, & EJ_{52} &= 2(u_{52}v_{52}-1), \\ E' &= \frac{4}{u_{52}^2(2u_{52}-1)^3v_{52}^2z} \left((\alpha_1+\alpha_2-1) - 2(2+\alpha_1+\alpha_2)u_{52} \right. \\ &\quad \left. + (1-\alpha_1-\alpha_2)u_{52}v_{52} + 4(1+\alpha_1)u_{52}^2v_{52} + 4(\alpha_2-\alpha_1)u_{52}^3v_{52} \right. \\ &\quad \left. - 2\alpha_1u_{52}^3v_{52}^2 + 4\alpha_1u_{52}^4v_{52}^2 \right). \end{aligned}$$

A.3.3 Resolution at b_5 The first chart is

$$\begin{aligned} u_{61} &= \frac{u_{32}}{v_{32}} = \frac{1}{uv}, & v_{61} &= v_{32} = u, \\ u &= v_{61}, & v &= \frac{1}{u_{61}v_{61}}, \\ J_{61} &= \frac{\partial u_{61}}{\partial u} \frac{\partial v_{61}}{\partial v} - \frac{\partial u_{61}}{\partial v} \frac{\partial v_{61}}{\partial u} = \frac{1}{uv^2} = u_{61}^2v_{61}. \end{aligned}$$

The exceptional line is $\mathcal{L}_6 : v_{61} = 0$. In this chart, the preimage of \mathcal{L}_3 is given by $u_{61} = 0$, and the preimage of \mathcal{L}_0 is not visible.

The flow is

$$\begin{aligned} u'_{61} &= \frac{1-u_{61}v_{61}^2}{v_{61}} + \frac{u_{61}(v_{61}+\alpha_1+\alpha_2u_{61}v_{61}^2)}{zv_{61}}, \\ v'_{61} &= -\frac{2+u_{61}v_{61}(2+v_{61})}{u_{61}} - \frac{v_{61}+2\alpha_1}{2z}, \end{aligned}$$

and contains a base point:

$$b_8 : u_{61} = -\frac{z}{\alpha_1}, v_{61} = 0.$$

The energy is given by

$$\begin{aligned} E &= -\frac{1 + 2u_{61}v_{61} + u_{61}^2v_{61}^2}{u_{61}^2v_{61}}, & EJ_{61} &= -(1 + 2u_{61}v_{61} + u_{61}^2v_{61}^2), \\ E' &= \frac{1}{2u_{61}^2v_{61}^2z} (2\alpha_1 + 3v_{61} + 4\alpha_1u_{61}v_{61} + 4(1 + \alpha_1 + \alpha_2)u_{61}v_{61}^2 \\ &\quad + 3u_{61}v_{61}^3 + 4\alpha_2u_{61}^2v_{61}^3 + 2\alpha_2u_{61}^2v_{61}^4). \end{aligned}$$

The second chart is

$$\begin{aligned} u_{62} &= u_{32} = \frac{1}{v}, & v_{62} &= \frac{v_{32}}{u_{32}} = uv, \\ u &= u_{62}v_{62}, & v &= \frac{1}{u_{62}}. \end{aligned}$$

The exceptional line is $\mathcal{L}_6 : u_{62} = 0$. In this chart, the preimages of \mathcal{L}_3 and \mathcal{L}_0 are not visible.

The flow is

$$\begin{aligned} u'_{62} &= -1 - 2u_{62} - 2u_{62}^2v_{62} + \frac{u_{62}(2\alpha_2u_{62} + 1)}{2z}, \\ v'_{62} &= \frac{v_{62}(-1 + u_{62}^2v_{62})}{u_{62}} - \frac{\alpha_1 + u_{62}v_{62}(1 + \alpha_2u_{62})}{zu_{62}}, \end{aligned}$$

and contains a base point is $u_{62} = 0, v_{62} = -\alpha_1/z$, which is b_8 .

The energy is given by

$$\begin{aligned} E &= -\frac{v_{62}}{u_{62}}(1 + 2u_{62} + u_{62}^2v_{62}), & EJ_{62} &= -v_{62}(1 + 2u_{62} + u_{62}^2v_{62}), \\ E' &= \frac{1}{2u_{62}^2z} (2\alpha_1 + 4\alpha_1u_{62} + 3u_{62}v_{62} + 4(1 + \alpha_1 + \alpha_2)u_{62}^2v_{62} \\ &\quad + 4\alpha_2u_{62}^3v_{62} + 3u_{62}^3v_{62}^2 + 2\alpha_2u_{62}^4v_{62}^2). \end{aligned}$$

A.4 Resolution at points b_6, b_7, b_8

A.4.1 *Resolution at b_6* The first chart is

$$\begin{aligned} u_{71} &= \frac{u_{42}}{v_{42} - \frac{\alpha_2}{z}} = \frac{z}{u(uvz - \alpha_2)}, & v_{71} &= v_{42} - \frac{\alpha_2}{z} = uv - \frac{\alpha_2}{z}, \\ u &= \frac{1}{u_{71}v_{71}}, & v &= u_{71}v_{71} \left(v_{71} + \frac{\alpha_2}{z} \right), \end{aligned}$$

which gives the Jacobian

$$\begin{aligned} J_{71} &= \frac{\partial u_{71}}{\partial u} \frac{\partial v_{71}}{\partial v} - \frac{\partial u_{71}}{\partial v} \frac{\partial v_{71}}{\partial u} = \frac{z}{u(\alpha_2 - uvz)} = -u_{71}, \\ J'_{71} &= -2u_{71} - 3u_{71}^2v_{71}^2 - \frac{u_{71}}{2z} (3 + 4(\alpha_1 + 2\alpha_2)u_{71}v_{71}) - \frac{(\alpha_1 + \alpha_2)\alpha_2}{z^2} u_{71}^2. \end{aligned}$$

The exceptional line is $\mathcal{L}_7 : v_{71} = 0$. In this chart, the preimage of \mathcal{L}_4 is given by equation $u_{71} = 0$, while the preimages of \mathcal{L}_1 and \mathcal{L}_0 are not visible.

The flow is given by

$$\begin{aligned} u'_{71} &= 2u_{71} + 3u_{71}^2 v_{71}^2 + \frac{u_{71}}{2z} (3 + 4(\alpha_1 + 2\alpha_2)u_{71}v_{71}) + \frac{(\alpha_1 + \alpha_2)\alpha_2}{z^2} u_{71}^2, \\ v'_{71} &= \frac{1}{u_{71}} - u_{71}v_{71}^3 - \frac{v_{71}}{z} (1 + (\alpha_1 + 2\alpha_2)u_{71}v_{71}) - \frac{\alpha_2(\alpha_1 + \alpha_2)u_{71}v_{71}}{z^2}, \end{aligned}$$

and contains no base points.

The energy is given by

$$\begin{aligned} E &= -\frac{1 + 2u_{71}v_{71} + u_{71}^2 v_{71}^3}{u_{71}} - \frac{\alpha_2}{u_{71}v_{71}z} (1 + 2u_{71}v_{71} + 2u_{71}^2 v_{71}^3) - \frac{\alpha_2^2}{z^2} u_{71}v_{71}, \\ EJ_{71} &= 1 + 2u_{71}v_{71} + u_{71}^2 v_{71}^3 + \frac{\alpha_2}{v_{71}z} (1 + 2u_{71}v_{71} + 2u_{71}^2 v_{71}^3) + \frac{\alpha_2^2}{z^2} u_{71}^2 v_{71}. \end{aligned}$$

The second chart is

$$\begin{aligned} u_{72} &= u_{42} = \frac{1}{u}, & v_{72} &= \frac{v_{42} - \frac{\alpha_2}{z}}{u_{42}} = \frac{u(uvz - \alpha_2)}{z}, \\ u &= \frac{1}{u_{72}}, & v &= \frac{u_{72}}{z} (zu_{72}v_{72} + \alpha_2). \end{aligned}$$

In this chart, the exceptional line \mathcal{L}_7 is given by equation $u_{72} = 0$, while the preimages of \mathcal{L}_4 , \mathcal{L}_1 , and \mathcal{L}_0 are not visible.

The flow

$$\begin{aligned} u'_{72} &= \frac{u_{72}}{2z} (1 + 2(\alpha_1 + 2\alpha_2)u_{72}) + 1 + 2u_{72} + 2u_{72}^3 v_{72}, \\ v'_{72} &= -\frac{2\alpha_2(\alpha_1 + \alpha_2) + 4(\alpha_1 + 2\alpha_2)u_{72}v_{72}z + v_{72}z(3 + 4z + 6u_{72}^2 v_{72}z)}{2z^2}, \end{aligned}$$

contains no base points.

A.4.2 *Resolution at b_7* The first chart is

$$\begin{aligned} u_{81} &= \frac{u_{51} - \frac{1 - \alpha_1 - \alpha_2}{8z}}{v_{51}} \\ &= \frac{u}{8(u+v)^3 z} ((\alpha_1 + \alpha_2 - 1)v^2 + (\alpha_1 + \alpha_2 - 1 + 4z)u^2 \\ &\quad + 2(\alpha_1 + \alpha_2 - 1 + 2z)uv + 8uz), \\ v_{81} &= v_{51} = \frac{u+v}{u}, \\ u &= \frac{8z}{v_{81}(8u_{81}v_{81}^2 z - (\alpha_1 + \alpha_2 - 1)v_{81} - 4z)}, \\ v &= \frac{8(v_{81} - 1)z}{v_{81}(8u_{81}v_{81}^2 z - (\alpha_1 + \alpha_2 - 1)v_{81} - 4z)}. \end{aligned}$$

In this chart, the exceptional line \mathcal{L}_8 is given by equation $v_{81} = 0$, while the preimages of \mathcal{L}_5 and \mathcal{L}_2 are not visible.

The flow is given by

$$\begin{aligned}
 u'_{81} = & \left(\frac{1 - 4\alpha_1 - 4\alpha_2}{2z} - 10 \right) u_{81} - \frac{\alpha_1(\alpha_1 + \alpha_2 - 1)^2}{64z^3} v_{81} - \frac{\alpha_1(\alpha_1 + \alpha_2 - 1)}{16z^2} \\
 & + \left(\frac{\alpha_1}{z} - \frac{5(\alpha_1 + \alpha_2 - 1)(\alpha_1 + \alpha_2)}{8z^2} \right) u_{81}v_{81} + \frac{(\alpha_1 + \alpha_2 - 1)^2(\alpha_1 + \alpha_2)}{32z^3} \\
 & + \frac{3\alpha_1(\alpha_1 + \alpha_2 - 1)}{8z^2} u_{81}v_{81}^2 + \frac{3(\alpha_1 + \alpha_2)}{z} u_{81}^2v_{81}^2 - \frac{2\alpha_1}{z} u_{81}^2v_{81}^3 \\
 & + \frac{3}{16z^2(8u_{81}v_{81}^2z - (\alpha_1 + \alpha_2 - 1)v_{81} - 4z)} \times (-(\alpha_1 + \alpha_2 - 1)^3 \\
 & - 4(\alpha_1 + \alpha_2 - 1)^2z - 32(3(\alpha_1 + \alpha_2 - 1) + 8z)z^2u_{81} \\
 & + 8(\alpha_1 + \alpha_2 - 1)(\alpha_1 + \alpha_2 - 1 + 4z)zu_{81}v_{81} + 512z^3u_{81}^2v_{81}), \\
 v'_{81} = & -4 + 4v_{81} - \frac{\alpha_1 + \alpha_2}{z} u_{81}v_{81}^3 + \frac{\alpha_1}{z} u_{81}v_{81}^4 + \frac{(\alpha_1 + \alpha_2 - 1)(\alpha_1 + \alpha_2)}{8z^2} v_{81}^2 \\
 & - \frac{\alpha_1(\alpha_1 + \alpha_2 - 1)}{8z^2} v_{81}^3 - \frac{\alpha_1}{2z} v_{81}^2 + \frac{\alpha_1 + \alpha_2}{2z} v_{81} \\
 & + \frac{24z(v_{81} - 1)}{8u_{81}v_{81}^2z - (\alpha_1 + \alpha_2 - 1)v_{81} - 4z}.
 \end{aligned}$$

There are no new base points.

The second chart is

$$\begin{aligned}
 u_{82} = & u_{51} - \frac{1 - \alpha_1 - \alpha_2}{8z} = \frac{u(u + v + 2)}{2(u + v)^2} - \frac{1 - \alpha_1 - \alpha_2}{8z}, \\
 v_{82} = & \frac{v_{51}}{u_{51} - \frac{1 - \alpha_1 - \alpha_2}{8z}} = \frac{u + v}{u} \left(\frac{u(u + v + 2)}{2(u + v)^2} - \frac{1 - \alpha_1 - \alpha_2}{8z} \right)^{-1}, \\
 u = & \frac{-8u_{82}^2z}{v_{82}(4u_{82}z + v_{82}(\alpha_1 + \alpha_2 - 1 - 8u_{82}z))}, \\
 v = & \frac{8u_{82}(u_{82} - v_{82})z}{v_{82}(4u_{82}z + v_{82}(\alpha_1 + \alpha_2 - 1 - 8u_{82}z))}.
 \end{aligned}$$

In this chart, the exceptional line \mathcal{L}_8 is given by equation $u_{82} = 0$, and the preimage of \mathcal{L}_5 by $v_{82} = 0$. The preimage of \mathcal{L}_2 is not visible.

The Jacobian is

$$J_{82} = \frac{v_{82}(4u_{82}z + v_{82}(\alpha_1 + \alpha_2 - 1 - 8u_{82}z))^3}{512u_{82}^3z^3},$$

while the derivative of the Jacobian is

$$\begin{aligned}
 J'_{82} = & \frac{\partial J_{82}}{u_{82}} u'_{82} + \frac{\partial J_{82}}{v_{82}} v'_{82} + \frac{\partial J_{82}}{z} \\
 = & -\frac{(4u_{82}z + v_{82}(\alpha_1 + \alpha_2 - 1 - 8u_{82}z))^2}{512u_{82}^4z^3} \times (3(\alpha_1 + \alpha_2 - 1)v_{82}^2u'_{82} \\
 & - 4u_{82}(u_{82}z + v_{82}(\alpha_1 + \alpha_2 - 1 - 8u_{82}z))v'_{82} \\
 & + 3(\alpha_1 + \alpha_2 - 1)v_{82}^2).
 \end{aligned}$$

The flow is given by

$$\begin{aligned}
 u'_{82} = & \frac{8}{v_{82}} - 6u_{82} - \frac{(\alpha_1 + \alpha_2 - 1)^2}{64z^3} u_{82} v_{82} (\alpha_1 u_{82} v_{82} - 2\alpha_1 - 2\alpha_2) \\
 & - \frac{1}{4z} (3(1 - \alpha_1 - \alpha_2) + 2(3\alpha_1 + 3\alpha_2 - 1)u_{82} - 2\alpha_1 u_{82}^2 v_{82} \\
 & \quad - 8(\alpha_1 + \alpha_2)u_{82}^3 v_{82} + 4\alpha_1 u_{82}^4 v_{82}^2) \\
 & + \frac{\alpha_1 + \alpha_2 - 1}{16z^2} (3(\alpha_1 + \alpha_2 - 1) - \alpha_1 u_{82} v_{82} - 8(\alpha_1 + \alpha_2)u_{82}^2 v_{82} \\
 & \quad + 4\alpha_1 u_{82}^3 v_{82}^2) \\
 & + 3 \frac{32z^2 + v_{82}(\alpha_1 + \alpha_2 - 1 + 4z)(\alpha_1 + \alpha_2 - 1 - 8u_{82}z)}{4v_{82}z(8u_{82}^2 v_{82}z - (\alpha_1 + \alpha_2 - 1)u_{82} v_{82} - 4z)}, \\
 v'_{82} = & 10v_{82} + \frac{(\alpha_1 + \alpha_2 - 1)^2 v_{82}^2 (\alpha_1 u_{82} v_{82} - 2\alpha_2 - 2\alpha_1)}{64z^3} \\
 & + \frac{(\alpha_1 + \alpha_2 - 1)v_{82}^2 (\alpha_1 + 10(\alpha_1 + \alpha_2)u_{82} - 6\alpha_1 u_{82}^2 v_{82})}{16z^2} \\
 & + \frac{v_{82}}{2z} (4\alpha_1 + 4\alpha_2 - 1 - 2\alpha_1 u_{82} v_{82} - 6(\alpha_1 + \alpha_2)u_{82}^2 v_{82} + 4\alpha_1 u_{82}^3 v_{82}^2) \\
 & + \frac{3v_{82}}{16z^2 ((1 - \alpha_1 - \alpha_2)u_{82} v_{82} - 4z + 8u_{82}^2 v_{82}z)} \times \\
 & \times ((\alpha_1 + \alpha_2 - 1)^3 v_{82} - 4(\alpha_1 + \alpha_2 - 1)^2 (2u_{82} - 1)v_{82}z \\
 & \quad - 32(\alpha_1 + \alpha_2 - 1)(u_{82} v_{82} - 3)z^2 + 256(1 - 2u_{82})z^3).
 \end{aligned}$$

There are no new base points.

The energy is given by

$$\begin{aligned}
 E = & - \frac{128(u_{82} v_{82} - 1)z^2(1 - \alpha_1 - \alpha_2 + 8u_{82}z)}{u_{82} v_{82} (8u_{82}^2 v_{82}z - (\alpha_1 + \alpha_2 - 1)u_{82} v_{82} - 4z)^3}, \\
 EJ_{82} = & - \frac{(u_{82} v_{82} - 1)(1 - \alpha_1 - \alpha_2 + 8u_{82}z)}{4u_{82}^4 z (8u_{82}^2 v_{82}z - (\alpha_1 + \alpha_2 - 1)u_{82} v_{82} - 4z)^3} \times \\
 & \times (4u_{82}z + v_{82}(\alpha_1 + \alpha_2 - 1 - 8u_{82}z))^3.
 \end{aligned}$$

A.4.3 *Resolution at b_8* The first chart is

$$\begin{aligned}
 u_{91} = & \frac{u_{62}}{v_{62} + \frac{\alpha_1}{z}} = \frac{z}{v(uvz + \alpha_1)}, \quad v_{91} = v_{62} + \frac{\alpha_1}{z} = uv + \frac{\alpha_1}{z}, \\
 u = & u_{91} v_{91} \left(v_{91} - \frac{\alpha_1}{z} \right), \quad v = \frac{1}{u_{91} v_{91}}, \\
 J_{91} = & \frac{\partial u_{91}}{\partial u} \frac{\partial v_{91}}{\partial v} - \frac{\partial u_{91}}{\partial v} \frac{\partial v_{91}}{\partial u} = \frac{z}{v(\alpha_1 + uvz)} = u_{91}, \\
 J'_{91} = & -2u_{91} - 3u_{91}^2 v_{91}^2 - \frac{\alpha_1(\alpha_1 + \alpha_2)u_{91}^2}{z^2} + \frac{3u_{91}}{2z} + \frac{2(2\alpha_1 + \alpha_2)u_{91}^2 v_{91}}{z}.
 \end{aligned}$$

The exceptional line is $\mathcal{L}_9 : v_{91} = 0$. In this chart, the preimage of \mathcal{L}_6 is given by equation $u_{91} = 0$, while the preimages of \mathcal{L}_3 and \mathcal{L}_0 are not visible.

The flow is

$$\begin{aligned} u'_{91} &= -2u_{91} - 3u_{91}^2 v_{91}^2 - \frac{\alpha_1(\alpha_1 + \alpha_2)u_{91}^2}{z^2} + \frac{3u_{91}}{2z} + \frac{2(2\alpha_1 + \alpha_2)u_{91}^2 v_{91}}{z}, \\ v'_{91} &= -\frac{1}{u_{91}} + u_{91} v_{91}^3 + \frac{\alpha_1(\alpha_1 + \alpha_2)u_{91} v_{91}}{z^2} - \frac{v_{91}}{z} - \frac{(2\alpha_1 + \alpha_2)u_{91} v_{91}^2}{z}, \end{aligned}$$

and contains no base points.

Energy:

$$\begin{aligned} E &= \frac{(v_{91}z - \alpha_1)(\alpha_1 u_{91}^2 v_{91}^2 - z - 2u_{91} v_{91} z - u_{91}^2 v_{91}^3 z)}{u_{91} v_{91} z^2}, \\ E J_{91} &= -1 - 2u_{91} v_{91} - u_{91}^2 v_{91}^3 - \frac{\alpha_1^2 u_{91}^2 v_{91}}{z^2} + \frac{2\alpha_1 u_{91}}{z} + \frac{\alpha_1}{v_{91} z} + \frac{2\alpha_1 u_{91}^2 v_{91}^2}{z}. \end{aligned}$$

The second chart is

$$\begin{aligned} u_{92} &= u_{62} = \frac{1}{v}, & v_{92} &= \frac{v_{62} + \frac{\alpha_1}{z}}{u_{62}} = \frac{v(uvz + \alpha_1)}{z}, \\ u &= u_{92}^2 v_{92} - \frac{\alpha_1}{z} u_{92}, & v &= \frac{1}{u_{92}}. \end{aligned}$$

The exceptional line is $\mathcal{L}_9 : u_{92} = 0$. In this chart, the preimages of \mathcal{L}_6 , \mathcal{L}_3 and \mathcal{L}_0 are not visible.

The flow is given by

$$\begin{aligned} u'_{92} &= -1 - 2u_{92} - 2u_{92}^3 v_{92} + \frac{u_{92}}{2z} + \frac{(2\alpha_1 + \alpha_2)u_{92}^2}{z}, \\ v'_{92} &= 2v_{92} + 3u_{92}^2 v_{92}^2 + \frac{\alpha_1(\alpha_1 + \alpha_2)}{z^2} - \frac{3v_{92}}{2z} - \frac{2(2\alpha_1 + \alpha_2)u_{92} v_{92}}{z}. \end{aligned}$$

and contains no base points.

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